Chapter 2

Existence of Nash networks in the One-way Flow Model of Network Formation

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Abstract

We study a one-way flow connections model of unilateral network formation. We prove the existence of Nash networks for games where the corresponding payoff functions allow for heterogeneity among the profits that agents gain by the network. Furthermore, we show by a counterexample that, when link costs are heterogeneous, Nash networks do not always exist.

Key Words: Non-cooperative games, network formation, Nash networks.

2.1 Introduction

Consider a group of agents who share certain profits by a network. In this network, the agents are represented as nodes. We consider one-way flow networks, where the links between the agents are directed and therefore depicted as arcs. The direction of the arcs corresponds to the flow of profits, i.e., a link between agents i and j which points at i means that i receives profits from being connected to j.

We study the formation of these one-way flow networks. We define a non-cooperative game in which agents have the opportunity to form costly links. Each agent can only form links pointing at him. All formed links together define the outcome network. We define a payoff function which assigns a payoff for each agent given the outcome network in the following

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way: each agent pays certain costs for each link that he formed and each agent gains certain profits from each other agent from whom a directed path to him exists in the outcome network. A network is called a Nash network if no agent can gain a stricty higher payoff by deviating from his set of formed links.

Our model is based on the one-way flow connections model proposed by [Bala and Goyal (2000a)]. They characterize and prove the existence of Nash networks for games where profits and link costs are homogeneous, i.e., all links are equally expensive and all agents have equal profits. [Galeotti (2006)] studies heterogeneity among profits and link costs and he characterizes the architecture of (strict) Nash networks for various settings while assuming such Nash networks exist.

In our paper we prove the existence of Nash networks for games with heterogeneous profits and owner-homogeneous link costs, i.e. all links have equal costs with respect to the agent who forms them. Furthermore, we provide a counterexample of a game with heterogeneous link costs for which Nash networks do not exist. The link costs of this game can be chosen arbitrarily close to the situation of owner-homogeneity.

Independently of us, and using a different approach, [Billand et al. (2007)] also proved the existence of Nash networks for games with heterogeneous profits and owner-homogeneous link costs. [Derks and Tennekes (2008b)] provide yet another alternative proof based directly on the ideas of [Billand et al. (2007)], but reducing the analysis to a short and elementary proof.

Several other models of network formation has been studied extensively in literature. Two-way flow models, i.e. models where profits can flow in both directions of a link, have been studied by [Bala and Goyal (2000a)], [Bala and Goyal (2000b)], [Galeotti et al. (2006)], and [Haller and Sarangi (2005)]. [Haller et al. (2007)] show the existence of Nash networks for two-way flow games with heterogeneous profits and homogeneous link costs and provide a counterexample with heterogeneous link costs where a Nash network does not exist. However, the results of [Haller et al. (2007)] do not imply ours, since they study two-way flow games while we focus on one-way flow games.

A model that is close to these one-way and two way flow models of network formation is the connections model introduced by [Jackson and Wolinsky (1996)]. Here, agents form links bilaterally instead of unilaterally. In other words, a link is only formed if both agents choose that link.

For an overview of literature on models of network formation we refer to [Jackson (2005)] and [Van den Nouweland (2005)].

2.2 Model and Notations

Let N denote a finite set of agents. We define a one-way flow network g on the agent set N as a set of links $g \subseteq N \times N$, where loops are not allowed, i.e. $(i, i) \notin g$ for all $i \in N$. A *path* from i to j in g is a sequence of distinct agents $i_1, i_2, \ldots, i_k$ with, $k \geq 1$, such that $i = i_1$,$j = i_k$ and $(i_s, i_{s+1}) \in g$ for each $s = 1, 2, \ldots, k-1$. Notice that for $k = 1$ we have that $i = i_1$ is a trivial path without links from i to himself.

Let $N_i(g) = \{j \in N \mid \text{a path from } j \text{ to } i \text{ exists in } g\}$ and let $N_i^d(g) = \{j \mid (j, i) \in g\}$. Note that $i \in N_i(g)$, and $i \notin N_i^d(g)$.

For each agent i, let $\pi_i : \mathcal{G}_N \rightarrow \mathbb{R}$ be a payoff function, where $\mathcal{G}_N$ is the set of all possible one-way flow networks on N. We will use the following payoff function, which has been proposed by [Galeotti (2006)].

$$\pi_i(g) = \sum_{j \in N_i(g)} v_{ij} - \sum_{j \in N_i^d(g)} c_{ij} \tag{2.1}$$

Here v_{ij} is the profit that agent i receives from being connected to j and c_{ij} is the cost of link (j, i) for agent i. The profits and costs are assumed to be non-negative throughout this paper.

We follow other literature on one-way flow models in the sense that the direction of the links indicates information flow. Consequently link $j \rightarrow i$, which is denoted by (j, i), is owned by agent i.

For convenience we will use the symbol '+' for the union of two networks as well as for the union of a network with a single link, e.g. $g \cup g' \cup \{(j, i)\}$ equals $g + g' + (j, i)$.

We say that link costs are *homogeneous* if there is a constant c with $c_{ij} = c$ for all $i, j \in N$. We say that link costs are *owner-homogeneous* if for each agent i there is a constant c_i with $c_{ij} = c_i$ for all $j \in N$. Otherwise, the link costs are *heterogeneous*.

In this paper we study a non-cooperative game. This game is played by the agents in N. Simultaneously and independently, each agent i chooses a, possibly empty, set S of agents he wants to connect to by creating the links (j, i), for each $j \in S$. Together, the links of all agents form a network $g \in \mathcal{G}_N$. Then, each agent i receives a payoff $\pi_i(g)$. Since each agent wants to maximize his payoff in response to what the other agents are doing, the focus of this paper is on Nash networks, i.e., networks in which no agent can profit from a unilateral deviation.

It is standard in literature to consider the set of agents, the costs, and the profits as fixed. However, our approach requires the comparison of different game situations. To facilitate this aproach we define a (non-cooperative) *network formation game* to be a triple (N, v, c) on agent set N with payoff functions π_i, $i \in N$, based on the profits $v = (v_{ij})_{i,j\in N}$ and costs $c = (c_{ij})_{i,j\in N}$, as described in Equation 2.1.

We define an *action* of agent i to a network g in network formation game (N, v, c) by a set of agents $S \subseteq N \setminus \{i\}$. The network that results after i chooses to link up with the agents in S, is described by

$$g_{-i} + \{(j, i) : j \in S\},$$

with g_{-i} denoting the network obtained from g after removing the links $(j, i) \in g$ owned by i. An action S^* of agent i is called a *best response* if

$$\pi_i\big(g_{-i} + \{(j, i) : j \in S^*\}\big) \geq \pi_i\big(g_{-i} + \{(j, i) : j \in S\}\big)$$

for all actions $S \subseteq N \setminus \{i\}$.

A network g is a Nash network in the game (N, v, c) if $N_i^d(g)$ is a best response for all $i \in N$, i.e., if for each agent i

$$\pi_i(g) \geq \pi_i\big(g_{-i} + \{(j, i) : j \in S\}\big)$$

for all actions $S \subseteq N \setminus \{i\}$.

Observe that for an agent i with costs sufficiently high, more specifically $c_{ik} > \sum_{j\in N, j\neq i} v_{ij}$ for all agents $k \neq i$, the only best response for agent i is $S = \emptyset$. Then, his payoff is v_{ii}, and any other action yields a smaller payoff (here we essentially need the fact that the profits are non-negative).

An agent k with no own links in a network g is only of interest for those agents i with $c_{ik} \leq v_{ik}$ since

$$\begin{aligned}
\pi_i(g + (k, i)) &= \sum_{j\in N_i(g+(k,i))} v_{ij} - \sum_{j\in N_i^d(g+(k,i))} c_{ij} \\
&= \sum_{j\in N_i(g)} v_{ij} + v_{ik} - \sum_{j\in N_i^d(g)} c_{ij} - c_{ik} \\
&= \pi_i(g) + v_{ik} - c_{ik}. \qquad (2.2)
\end{aligned}$$

2.3 Owner-homogeneous Costs

In this section we will prove existence of Nash networks in network formation games where the costs are owner-homogeneous.

For costs sufficiently small, the so-called *cycle* networks are Nash networks. Cycle networks consist of one cycle joining all agents (see Figure 2.1).

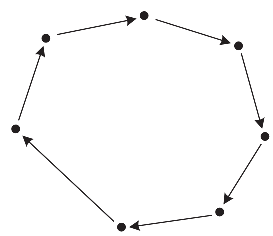

Fig. 2.1 Cycle network

Lemma 2.1. *For an owner-homogeneous cost network formation game (N, v, c), with $c_i \leq \sum_{j \in N, j \neq i} v_{ij}$ for all $i \in N$, all cycle networks are Nash networks.*

Proof. Without loss of generality, consider $N = \{1, 2, \ldots, n\}$ and the cycle network $g = \big\{(i, i+1) : i = 1, \ldots, n-1\big\} + (n, 1)$.

Any agent i obtains $\pi_i(g) = \sum_{j \in N} v_{ij} - c_i$, and there is no other network with a larger payoff, implying that g is a Nash network. □

In the owner-homogeneous costs case, we also observe the following: if link (j, k) is present in g, then linking up with agent k is at least as good for an agent $i \neq j, k$, as linking up with j:

$$\pi_i(g + (k, i)) \geq \pi_i(g + (j, i)) \quad \text{whenever } (j, k) \in g. \tag{2.3}$$

In the next theorem, we prove the existence of Nash networks for games with owner-homogeneous link costs. This proof is constructive in nature. Either any cycle network constitutes a Nash network or there is an agent that is not interested in being 'involved'. In the latter case, there might be an agent i who is interested in linking up with this uninvolved agent. In that case the profit values are adapted as described in the proof, and a Nash network is searched in the situation without the uninvolved agent; next, this network is extended by connecting the uninvolved agent with agent i.

The case where no agent is interested in linking up with this uninvolved agent is somewhat simpler, as any Nash network on the set of agents excluding the uninvolved agent is also a Nash network on the full set of agents.

Theorem 2.1. *Nash networks exist for any network formation game with owner-homogeneous costs.*

Proof. We will prove the theorem by induction. 1-agent network formation games trivially have a Nash network. Suppose that (N, v, c) with $N = \{1, 2, \dots, n\}$ is an owner-homogeneous cost network formation game that does not have a Nash network, while all network formation games with less than n agents do have Nash networks. According to Lemma 2.1, this implies that there is an agent i with $c_i > \sum_{j \in N, j \neq i} v_{ij}$.

Without loss of generality assume $i = n$. Observe that the best response of agent n in any network is the empty set. Consider the owner-homogeneous cost network formation game (N', v', c'), with $N' = N \backslash \{n\}$, and v' and c' equal to v and c restricted to the agents in N'. Let π'_i denote the payoff function for agent i in (N', v', c'). It is clear that $\pi'_i(g) = \pi_i(g)$ for each network g on N', and $i \neq n$.

Since N' has $n-1$ agents, (N', v', c') has a Nash network, say g'. Consider g' as a network on N, and recall the assumption that (N, v, c) does not have a Nash network. Therefore, there is an agent i in (N, v, c) who does not play his best response in g'. Of course $i \neq n$, as $N_n^d(g') = \emptyset$. Let $T \subseteq N \setminus \{i\}$ be a best response of i in g', and suppose $n \notin T$. Then $(g')_{-i} + \{(j, i) : j \in T\}$ is a network in N' so that

$$\begin{aligned}
\pi'_i(g') &= \pi_i(g') \\
&< \pi_i\big((g')_{-i} + \{(j, i) : j \in T\}\big) \\
&= \pi'_i\big((g')_{-i} + \{(j, i) : j \in T\}\big),
\end{aligned}$$

which is a contradiction with g' being a Nash network for (N', v', c').

Now suppose that $n \in T$. Without loss of generality assume $i = 1$. From $n \in T$ and $N_n^d(g') = \emptyset$ we conclude that $c_1 \leq v_{1n}$ must hold (see 2.2). Consider the following adapted profits $v^* = (v^*_{ij})_{i,j \in N'}$:

$$v^*_{ij} = \begin{cases} v_{ij} & \text{if } j \neq 1, \\ v_{i1} + v_{in} & \text{if } i \neq 1,\ j = 1, \\ v_{11} + v_{1n} - c_1 & \text{if } i, j = 1. \end{cases}$$

Observe that these values are non-negative. Let π^*_i denote the payoff functions in (N', v^*, c'). The profits v^*_{ij} are chosen such that $\pi^*_i(g) = \pi_i(g + (n, 1))$ holds for all networks g on N', and for all $i \in N'$.

By assumption, the network formation game (N', v^*, c') has a Nash network, say g^*; since (N, v, c) does not have a Nash network, there is an agent i in N who can improve in the network $g^* + (n, 1)$, in the context of the game (N, v, c), say by choosing the links with the agents in $S \subseteq N\backslash\{i\}$. This agent is not n because $N_n^d(g^* + (n, 1)) = \emptyset$.

Suppose $i \neq 1$. If $n \in S$, then according to (2.3), the action $S\backslash\{n\} \cup \{1\}$ is at least as good as S; therefore we may assume $n \notin S$. The resulting network $(g^* + (n, 1))_{-i} + \{(j, i) : j \in S\}$, after i performs the improvement, yields a higher payoff for agent i. Then

$$\begin{aligned}
\pi_i^*(g^*) &= \pi_i(g^* + (n,1)) \\
&< \pi_i\big((g^* + (n,1))_{-i} + \{(j,i) :\ j \in S\}\big) \\
&= \pi_i\big((g^*_{-i} + \{(j,i) :\ j \in S\}) + (n,1)\big) \\
&= \pi_i^*(g^*_{-i} + \{(j,i) :\ j \in S\}) \\
&\leq \pi_i^*(g^*),
\end{aligned}$$

where the latter inequality holds because of g^* being a Nash network for (N', v^*, c'). Thus, we arrived at a contradiction, so that we must have $i = 1$.

Due to $c_1 \leq v_{1n}$, and agent n having no own links, we may assume $n \in S$ (see the observation concerning (2.2)). Then

$$\begin{aligned}
\pi_1^*(g^*) &= \pi_1(g^* + (n,1)) \\
&< \pi_1(g^*_{-1} + \{(j,1) :\ j \in S\}) \\
&= \pi_1\big(g^*_{-1} + \big\{(j,1) :\ j \in S\backslash\{n\}\big\} + (n,1)\big) \\
&= \pi_1^*\big(g^*_{-1} + \big\{(j,1) :\ j \in S\backslash\{n\}\big\}\big) \\
&\leq \pi_1^*(g^*);
\end{aligned}$$

a contradiction. We conclude that (N, v, c) must have a Nash network. □

Observe that the Nash networks we obtain, have at most one cycle, in case the Nash networks excluding the uninvolved agent also have at most one cycle. The same applies when considering networks where agents have an outdegree of at most one. The following corollary is now easily established.

Corollary 2.1. *Nash networks exist, with at most one cycle and maximum outdegree of at most 1, for any network formation game with owner-homogeneous costs.*

There may also exist Nash networks with multiple cycles and with outdegrees higher than 1. Consider the following example.

Example 2.1. Let $n = 7$, and let $c_{ij} = 1$ and $v_{ij} = 1$ for all $i, j \in N$. Consider the network with two cycles that is depicted in Figure 2.2. Notice that agent i has two outgoing links. It can be verified that this network is a Nash network. We revisit this network in section 2.5.

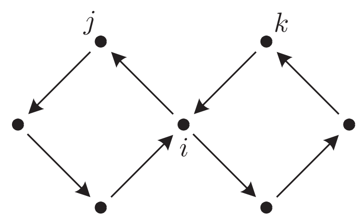

Fig. 2.2 A Nash network with two cycles

2.4 Heterogeneous Costs

For network formation games with heterogeneous costs, Nash networks do not always exist as we will see by the next example. The link costs in this example can be chosen arbitrarily close to the situation of owner-homogeneity.

Example 2.2. Consider a network formation game (N, v, c) where $N = \{1, 2, 3, 4\}$, where the profits are owner-homogeneous and normalized to 1 (i.e., $v_{ij} = 1$ for all i, j), and where the costs are heterogeneous. The numbers next to the links in Figure 2.3 indicate the costs of these links.

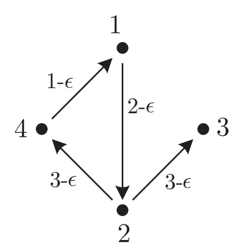

Fig. 2.3 The link costs

Here, ϵ is a strictly positive number which can be chosen arbitrarily close to 0. The costs of the links that are not depicted in this figure are the following:

- links directed to agent 1 have costs $1 + \epsilon$,
- links directed to agent 2 have costs $2 + \epsilon$,
- links directed to agents 3 and 4 have costs $3 + \epsilon$,

The best response of agent 4 to any network is either $\{2\}$ or $\emptyset$, since those are the only actions for which agent 4 has a non-negative payoff. First, suppose that agent 4 plays $\{2\}$ as a best response in a Nash network.

Consequently, the unique best response of agent 1 is $\{4\}$. Agent 2 has one unique best response to this situation: $\{1\}$. Finally, agent 3 has one unique best response, which is $\{2\}$. The obtained network is the same as depicted in Figure 2.3. It follows that $\{2\}$ is not a best response of agent 4, since $\emptyset$ gives a higher payoff. This contradicts our assumption. Hence, there is no Nash network in which agent 4 plays $\{2\}$.

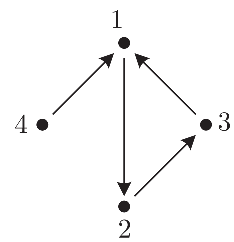

Fig. 2.4 Network obtained in Example 2.2

Now suppose that agent 4 plays $\emptyset$ as a best response in a Nash network. Agent 1 will include 4 in every best response to this situation. Then, the unique best response for agent 2 is $\{1\}$. To this situation, the unique best response of agent 3 is $\{2\}$. Hence, the unique best response of agent 1 is $\{3, 4\}$ (see Figure 2.4). Now agent 4 has a unique best response to this new situation, which is $\{2\}$. This contradicts our assumption of agent 4 playing $\emptyset$ in a Nash network. Hence no Nash networks exist.

As is clear from this example, Nash networks do not always exist for network formation games with heterogeneous costs. However, some of the arguments in the proof of our main result (Theorem 2.1) can be generalized to hold also for a specific class of payoff functions with heterogeneous costs. For example, expression (2.3) is also fulfilled when the following conditions are met:

$$c_{ij} \geq c_{ik} - v_{ik} \quad \text{for all different agents } i, j, k.$$

Unfortunately, not all arguments have their counterpart in the heterogeneous case; especially the relation between the Nash networks and cycles is not apparent. Therefore, a generalization of our main result needs a different approach, and in our subsequent article [Derks et al. (2008)] we show the existence of Nash networks for a more general class of network formation games. The adapted proof is also constructive in nature, and starts with a framework of properties that the payoff functions have to obey. These properties provide a generalization of our main result, which is further discussed in [Derks and Tennekes (2008a)].

2.5 Strict Nash Networks

A network g is a *strict-Nash network* if $N_i^d(g)$ is a unique best response for each agent i. [Galeotti (2006)] studies the architecture of strict-Nash networks in detail. He shows that the maximum outdegree of strict-Nash networks is at most 1 in network formation games with (owner-)homogeneous link costs. This is confirmed by Example 2.1. The network depicted in Figure 2.2 is not strict-Nash, because agent j can deviate by forming (k, j) instead of (i, j) which gives him the same payoff due to link costs owner-homogeneity. Notice that in the newly obtained network, agent i can deviate by removing link (k, i) which gives him a higher payoff. The cycle network that we now obtain is both Nash and strict-Nash.

The following example shows that strict-Nash networks do not always exist for games with owner-homogeneous link costs.

Example 2.3. Let again $N = \{1, \ldots, n\}$ be the set of agents. Let $c_1 = n - 1$, and $c_i = 1$ for all $i \neq 1$. Let $v_{ij} = 1$ for all $i, j \in N$.

It is easily seen that in each strict-Nash network, all agents in $N \setminus \{1\}$ are contained in one cycle.

Either agent 1 is also contained in this cycle or not. Suppose he is. Then by his one link he receives $n-1$ profits, and the link itself costs $n-1$. Hence, he is indifferent about maintaining this link. Thus, a cycle network cannot be strict-Nash.

Now suppose that agent 1 is not contained in the cycle on $N \setminus \{1\}$. Then, by forming a link with one of the other agents, agent 1 receives $n-1$ profits, and pays $n - 1$. Therefore, he is indifferent about forming such a link. Hence, again the network cannot be strict-Nash. Therefore, we conclude that strict-Nash networks do not exist for this game.

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