Local Dynamics in Network Formation

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Abstract

We study a dynamic network formation game in which agents play local actions, which are adding, removing or replacing a link. We prove that finding a best local response can be done in polynomial time, while the problem of finding a best global response is $\mathcal{NP}$-hard. We show that for general classes of payoff functions, which are based on axiomatic properties, local-Nash and global-Nash networks exist. Also, we show that the dynamic process of iterated local actions always terminates at a global-Nash network.

Keywords: non-cooperative games, network formation, Nash networks.
JEL classification: C72, D85

1 Introduction

In this paper we study the formation of endogenous networks. Social and economic networks are typically endogenous, since they are shaped by autonomous agents (e.g. persons or organizations) who correspond to the nodes. Via pairwise links, these agents are able to share valuable information.

In this paper we propose a dynamic model of network formation where agents form links unilaterally. Our model is based on Bala and Goyal (2000a). For a brief introduction and overview of literature on other models of network formation we refer to Jackson (2005) and to Van den Nouweland (2005).

Bala and Goyal (2000a) model network formation as a non-cooperative game. Here, an agent's action is defined as a set of links. The links of all agents together define a directed network. The links that are formed by agent i are depicted by arcs pointing at i. A payoff function assigns a payoff for each agent on base of the formed network.

The payoff functions that Bala and Goyal (2000a) study are the following. Each agent pays a certain cost for each own link, i.e. for each link pointing at him. Further, each agent receives certain profits from being connected to each other agent. Here, two cases are considered. In the first case, agent i is connected to agent j if a directed path exists from j to i, and in the second case if an undirected path exists between them. These two cases are called the one-way flow and the two-way flow model respectively. These models have been extended by allowing

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heterogeneity among link costs and profits by Galeotti (2006) for the one-way flow case and by Galeotti et al. (2006) for the two-way flow case. Further, both models have been extended by introducing decay (Bala and Goyal (2000a), Galeotti (2006), and Galeotti et al. (2006)), i.e. the the amount of profits depend on the length of the connection path counted in number of links. Another extension that has been studied in literature is that links are not perfectly reliable (Bala and Goyal (2000b), Haller and Sarangi (2005), and Haller et al. (2007)).

Observe that all these models (one-way and two-way flow, with and without decay, with perfectly and imperfectly reliable links) only differ from each other in payoff function. In this paper we prove the existence of Nash networks for general classes of payoff functions. For this purpose, we propose a framework of axiomatic payoff properties that is inspired by the one-way flow model without decay. These properties are intuitive and they are sufficient to guarantee the existence of Nash networks.

Our payoff properties, and our proofs are oriented on local actions, which consist of the following types: an addition of a link, a deletion of a link, a replacement of a link, and a pass. Local actions are easier to analyze than global actions (defined as changing the whole set of own links at once). From a computational point of view, we show that that the problem of finding a best global response is $\mathcal{NP}$-hard, while finding a best local response can be done in polynomial time. We use this local approach in order to obtain global results, which are the existence of global-Nash network, and the termination of an iterative procedure of local actions at a global-Nash network.

We define a local-Nash network as a network in which no agent can improve by a unilateral local action. We prove the existence of local-Nash networks for a framework of payoff properties. Moreover, the networks that we find have a specific architecture that is characterized by the property that each agent has at most one outgoing link. We show that local-Nash networks with this architectural property are also global-Nash when the payoff function satisfies an additional property. We show independence of our properties, and moreover, we show that this framework of properties yields a generalization with respect to the payoff function in one-way flow model with owner-homogeneous link costs and heterogeneous profits.

Further, we study an iterative procedure of good local responses. This procedure starts with an arbitrary initial network. At each stage, an agent is selected at random, where each agent has a fixed, stage-independent probability to be selected. This agent plays a good local response. The procedure terminates at a specific network when this network is reached, and moreover, when no agent wants to play a good local response other than pass. We prove that our procedure terminates at a local-Nash network which is also global-Nash, whenever the corresponding payoff function satisfies all our properties.

The described dynamic procedure resembles the one described by Bala and Goyal (2000a). However, they examine only global actions, and multiple agents may play simultaneously during each stage of their model. Watts (2001) and Jackson and Watts (2002) also study dynamic models of network formation, but their models are based on the bilateral model of network formation introduced by Jackson and Wolinsky (1996).

The outline of this paper is as follows. In section 2 we present the model and the notations that we use throughout. In section 3 we study the complexity of determining best global and best local responses. Here, we show that the problem of finding a best global response is $\mathcal{NP}$-hard, while the problem of finding a best local response can be solved in polynomial time. In section 4 we prove the existence of local-Nash and global-Nash networks for games where the payoff function satisfies a specific set of axiomatic properties. Then, we prove independence of these properties and relate them to payoff functions of the one-way flow model. In section 5, we study the play of our dynamic game. We provide an iterative procedure of good local responses, and show that it terminates at a global-Nash network whenever the payoff function satisfies our payoff properties. Finally, in section 6 we provide concluding remarks.

2 Model and notations

In this section we provide our model of network formation and we introduce the notations that we will use throughout this paper.

2.1 Network

Let N denote a finite fixed set of agents. We define a *network* g on the agent set N as a set of links $g \subseteq N \times N$, where loops are not allowed, i.e. $(i,i) \notin g$ for all $i \in N$. Let $\mathcal{G}$ be the set of all possible networks on N. A *directed path* from i to j in g is a sequence of distinct agents $i_1, i_2, \ldots, i_k$ with, $k \geq 1$, such that $i = i_1$, $j = i_k$ and $(i_s, i_{s+1}) \in g$ for each $s = 1, 2, \ldots, k-1$. Notice that for $k = 1$ we have that $i = i_1$ is a trivial directed path without links from i to himself. An *undirected path* is defined analogously where either (i_s, i_{s+1}) or (i_{s+1}, i_s) is contained in g for each $s = 1, 2, \ldots, k-1$. Further, a *directed cycle* and an *undirected cycle* are defined in the same way with the exception that $i_1 = i_k$.

For convenience we will use use the symbols '+' and '−' for the union, respectively the set exclusion of two networks, or for a network and a single link. In case of ambiguity, these operations are applied from left to right. For instance, the notation $g - g' + (j,i)$ equals $(g \setminus g') \cup \{(j,i)\}$.

Let $\mathrm{Car}(g)$, the carrier of g, denote the set of so-called *active* agents in the network g, being those agents who are begin- or endpoints of a link in g. For a network g we define g^j, the *component* of g that contains agent j, as the network containing all links that are connected to j by some undirected path.

We say that a link (j,i), which is directed to i, is *owned* by i. Let $N_i(g) = \{j \in N :$ a directed path from j to i exists in $g\}$ be the set of agents who are *observed* by i in g, and let $N_i^d(g) = \{j : (j,i) \in g\}$ be the set of *neighbors* of i in g. Note that $i \in N_i(g)$ and $i \notin N_i^d(g)$.

Let g_{-i} denote the network obtained from g after removing the links owned by i. Notice that an outgoing link of i, e.g. (i,j), may still exist in g_{-i}. Further, we define $g_{-ij} = g_{-i}^j + (j,i)$, where g_{-i}^j means $(g_{-i})^j$, i.e. the component of g_{-i} with j being active. We will come back to this definition and the related definition *beneficiality* in section 4.

For each agent i, let $\pi_i : \mathcal{G} \to \mathbb{R}$ be a payoff function. In section 4 we introduce axiomatic properties for payoff functions in general. These properties are inspired by the payoff functions of the one-way flow model studied by Bala and Goyal (2000a) and Galeotti (2006). They consider the following class of payoff functions:

$$\pi_i(g) = \sum_{j \in N_i(g)} v_{ij} - \sum_{j \in N_i^d(g)} c_{ij} \tag{1}$$

for constants $(v_{ij})_{i,j \in N}$ and $(c_{ij})_{i,j \in N, i \neq j}$. Here, v_{ij} is intepreted as the profit that agent i receives from being connected to j and c_{ij} is interpreted as the cost of link (j,i) for agent i. All profits and costs are non-negative. We say that link costs are *homogeneous* if there is a constant c with $c_{ij} = c$ for all $i, j \in N$, $i \neq j$. We say that link costs are *owner-homogeneous* if for each agent i there is a constant c_i with $c_{ij} = c_i$ for all $j \in N \setminus \{i\}$. Otherwise, link costs are *heterogeneous*. These definitions also apply to the profits. We will refer to payoff functions defined by (1) as B&G functions.

2.2 Network formation game

Given a set of agents N and a payoff function π_i for each agent i, a network formation game proceeds in stages $1, 2, 3, \ldots$. Let g_t be the network at the beginning of stage t, which is known to all agents. The initial network g_1 can be any network in $\mathcal{G}$. Then, at stage t according to a probability device, an agent, say i, is selected. We assume that at each stage all agents have positive (stage independent) probabilities of being selected. Now, stage t proceeds by allowing one agent to modify the network g_t by adjusting his set of links. Thus, a new network g_{t+1} results, which marks the start of stage $t+1$. The game ends with network g_* if no agent wants to adjust his links. In that case, each agent i receives payoff $\pi_i(g_*)$.

As for the stage adjustments we distinguish two cases: one of local adjustments and one of global adjustments. In the first case the actions of agent i are restricted to (1) passing, (2) adding a new link pointing at i, (3) deleting a link pointing at i, or (4) a replacement, which is a combination of the previous two. These four types of actions are called *local actions*. In the second case agent i is allowed to completely change the set of links pointing at him. These actions are called *global actions*.

Formally, we define an action of agent i as a set of agents $S \subseteq N \setminus \{i\}$. For a global action, there are no restrictions on S. For a local action we require $|N_i^d(g) \setminus S| \leq 1$ and $|S \setminus N_i^d(g)| \leq 1$. The network, after i chooses to link with the agents in S, is described by

$$g_{-i} + \{(j, i) : j \in S\}.$$

A local action S of agent i is called a *good local response* if

$$\pi_i\big(g_{-i} + \{(j, i) : j \in S\}\big) \geq \pi_i(g).$$

A local action S of agent i is called a *best local response* if

$$\pi_i\big(g_{-i} + \{(j, i) : j \in S\}\big) \geq \pi_i\big(g_{-i} + \{(j, i) : j \in T\}\big),$$

for all local actions T. A network g is called a *local-Nash network* if $N_i^d(g)$ is a best local response for all $i \in N$. A network g is called a *strict local-Nash network* if $N_i^d(g)$ is the unique best local response for all $i \in N$. Analogous definitions apply for the global case.

The following example with three agents illustrates how the dynamic formation game is played.

Example 1 *Let the set of agents be $N = \{1, 2, 3\}$ and for each agent $i \in N$, let π_i be a B&G function as described in (1), where $v_{ij} = 2$ and $c_{ij} = 1$ for all agents j.*

Let the initial network in this example, g_1, be the empty network and let the agents play local actions. The play of the game is shown in Table 1. In the second last column the selected agent is given, but the agents do not know the order in which they are chosen in advance. The corresponding networks are depicted in Figure 1. Notice that all played local actions in this example are best local responses. Furthermore, the final network, g_5, is strict local- and global-Nash.

Table 1: Play of the game.

Stage t	Network g_t	$\pi_1(g_t)$	$\pi_2(g_t)$	$\pi_3(g_t)$	Selected agent	Local action
1	$g_1 = \emptyset$	0	0	0	1	add $(3,1)$
2	$g_2 = \{(3,1)\}$	1	0	0	3	add $(1,3)$
3	$g_3 = \{(3,1),(1,3)\}$	1	0	1	2	add $(3,2)$
4	$g_4 = \{(3,1),(1,3),(3,2)\}$	1	3	1	1	replace $(3,1)$ with $(2,1)$
5	$g_5 = \{(2,1),(1,3),(3,2)\}$	3	3	3	All agents pass.	

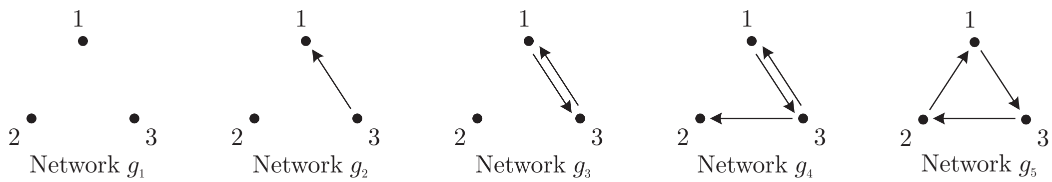

Figure 1: The networks obtained in Example 1

3 Best response

We motivate our local approach by the fact that local actions are easier to deal with than global actions. In this section, we show that finding a best local response is polynomial bounded in n (the number of agent), while the problem of finding a best global response is $\mathcal{NP}$-hard, even for a relatively small class of payoff functions: B&G functions with homogeneous link costs. Baron et al. (2008) also study the computational complexity of finding best responses in unilateral network formation games, but they consider a different class of payoff functions.

First observe that there are 2^{n-1} possible global actions that an agent can perform. In the following theorem we show that the number of local actions that an agent can perform, is bounded by the square of n.

Theorem 2 *The number of possible local actions that an agent can perform, is bounded by n^2, where n is the number of agents.*

Proof. The number of possible local actions that agent i can perform, depends on the number of neighbors of i in g. If we denote this number by $m \leq n-1$, then agent i can do $n-m-1$ additions, m deletions, and $(n-m-1)m$ replacements. Hence, the number of possible local actions for agent i equals

$$\begin{aligned} n-m-1+m+(n-m-1)m &= (n-1)+(n-m-1)m \\ &\leq (n-1)+(n-m-1)(n-1) \\ &= (n-1)(n-m) \\ &\leq n^2. \end{aligned}$$

□

Thus, finding a best local response can be done within polynomial time.

The complexity of finding a best global response is dependent on the payoff function. Here, we restrict to a relatively small class of payoff functions, namely B&G functions with homogeneous link costs. For this class, we show that the *Best Global Response Problem* (*BGRP* in short) is $\mathcal{NP}$-hard.

Theorem 3 *Let π be a B&G function. Then BGRP is $\mathcal{NP}$-hard, even when link costs are homogeneous.*

Proof. We prove this by reduction from the Minimum Set Cover problem (MSC), which is a well-known $\mathcal{NP}$-hard problem (see Karp (1972)). Let $\mathcal{K} = \{K_1, K_2, \ldots, K_k\}$ be a collection of k subsets of a finite set $X = \{1, 2, \ldots, x\}$ such that $X \subseteq \bigcup_{j=1}^{k} K_j$. MSC is the problem of finding a subset $\mathcal{K}' \subseteq \mathcal{K}$ of minimum cardinality such that every element in X belongs to at least one member of $\mathcal{K}'$. Notice that such a set cover $\mathcal{K}'$ exists.

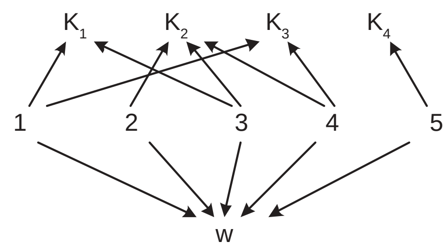

Figure 2: Network g

Next we show how to reduce MSC to BGRP. Let the agent set be initiated as $N = \{1, \ldots, x, K_1, K_2, \ldots, K_k, w, y\}$. Let $g \in \mathcal{G}$ be the network on N, built up as follows: for each agent $i \in X$ we create a link (i, w) and for each agent $i \in K_j$ we create a link (i, K_j). In Figure 2 an example of such a network is shown. Let the profits of agent y have the following values:

$$v_{yi} = \begin{cases} 1 & \text{if } i \in X; \\ 1 - \frac{1}{2k} & \text{if } i \in \mathcal{K}; \\ 0 & \text{if } i \in \{w, y\}. \end{cases}$$

Let the link costs be homogeneous; let $c = 1$.

We show that the problem of finding a best global response S for agent y with respect to g is equivalent to the problem of finding a minimum subset of $\mathcal{K}$ that covers X.

Observe that we may restrict to $S \subseteq \{K_1, \ldots, K_k, w\}$, because every $i \in X$ is an element of some K_j, and therefore agent y would receive at least as much payoff from replacing i by K_j. Further, if $w \in S$, then $S = \{w\}$, since the cost of any additional link exceeds the extra profits. Hence, either $S = \{w\}$ or $S \subseteq \mathcal{K}$. Observe that the action $\{w\}$ yields the payoff $x - 1$ for agent y.

Let $\mathcal{K}' \subseteq \mathcal{K}$ and let T be an action defined as $T = \mathcal{K}'$. Then T yields the following payoff for agent y:

$$\pi_y(g_{-y} + \{(i, y) : i \in T\}) = k'(1 - \frac{1}{2k}) + t - k' = -\frac{k'}{2k} + t$$

where $k' = |\mathcal{K}'|$ and t is the number of members of X that are covered by $\mathcal{K}'$.

If $\mathcal{K}'$ does not cover X, then $t \leq x - 1$, and hence $-\frac{k'}{2k} + t \leq -\frac{k'}{2k} + x - 1 < x - 1$. In other words, the action T yields a payoff which is strictly less than the payoff $x-1$ which corresponds to the action $\{w\}$. Hence we conclude that if $\mathcal{K}'$ does not cover X, then the corresponding action T is not a best global response.

If $\mathcal{K}' \subseteq \mathcal{K}$ covers X, then $t = x$, and hence $-\frac{k'}{2k} + t = -\frac{k'}{2k} + x > x - 1$. Thus, the action T yields a strictly higher payoff than the payoff $x - 1$ which corresponds to the action $\{w\}$. So every action that is a set cover yields a strictly higher payoff than the payoff from the action $\{w\}$. Of all actions that are set covers, the ones with the lowest cardinality are best global responses, because the payoff $\frac{-k'}{2k} + x$ is maximal if k' is minimal. We therefore conclude that each best global response of agent y with respect to network g corresponds to a minimum set cover.

Since the transformation from any MSC instance to a BGRP instance can be done in polynomial time and since MSC is $\mathcal{NP}$-hard (see Karp (1972)), it follows that BGRP is also $\mathcal{NP}$-hard. □

Observe that BGRP can be interpreted as the problem of maximizing a set function, since the playing agent i and the network g_{-i} are fixed in the BGRP. Hence we define a specific set function $f : 2^{N\setminus\{i\}} \to \mathbb{R}$ as

$$f(S) = \pi_i(g_{-i} + \{(j,i) : j \in S\})$$

for each $S \subseteq N \setminus \{i\}$, where network g and agent i are fixed.

A set function f is called *submodular* if

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \text{ for all } S, T \subseteq N \setminus \{i\},$$

and *supermodular* if the left-hand side is less than or equal to the right-hand side.

For maximizing supermodular set functions in general, which is equivalent to minimizing submodular set functions, Grötschel et al. (1981) proposed a polynomial-time algorithm. Alternative algorithms that are more efficient in practice are proposed independently by Schrijver (2000) and Iwata et al. (2001).

Garey and Johnson (1979) show that the problem of maximizing submodular set functions is $\mathcal{NP}$-hard, due to the fact that it is a general case of the max-cut problem. The problem of maximizing submodular set functions has also been studied by Nemhauser et al. (1978), Lovasz (1983), and Lee et al. (1996), among others.

It can be shown that f is submodular whenever the corresponding payoff function π is a B&G function, due to the fact that for disjoint actions S and T, the sets $N_i(g_{-i}+\{(j,i) : j \in S\})$ and $N_i(g_{-i}+\{(j,i) : j \in T\})$ may intersect. Hence, the BGRP is a special case of the $\mathcal{NP}$-hard problem of maximizing a submodular set function. In Theorem 3 we have shown that even this special case is $\mathcal{NP}$-hard.

4 Nash networks

In this section we study the existence of Nash networks. Bala and Goyal (2000a) show that global-Nash networks exist when payoff functions are B&G functions with homogeneous link costs and profits. For B&G functions with owner-homogeneous link costs, i.e. $c_{ij} = c_i$, and heterogeneous profits, the existence of Nash networks has been proved independently by Derks et al. (2008) and Billand et al. (2008). Derks and Tennekes (2008b) provide an alternative and easy accessible proof that is based on an idea of Billand et al. (2008). By means of a counterexample, Derks et al. (2008) show that Nash networks may fail to exist for the heterogeneous link costs case, even if link costs are arbitrarily 'close' to the situation of owner-homogeneity, i.e. $|c_{ij} - c_{ik}| \leq \epsilon$, for an arbitrarily small $\epsilon > 0$, and for all i, j, and k.

In this section we prove that global-Nash networks exist for a class of payoff functions that is defined by a framework of axiomatic payoff properties. These properties and proofs are oriented locally. First, in subsection 4.1, we prove that local-Nash networks with some specific architecture are also global-Nash when the payoff function satisfies three of our properties. Then, in subsection 4.2, we show the existence of local-Nash networks when the payoff function satisfies a specified set of properties. The local-Nash networks that we find are also global-Nash when combining these properties. We choose the properties in such a way that they are intuitive and such that they allow us to proceed with our line of proof in the most general way. In subsection 4.3, we show that the properties are independent of each other, and in subsection 4.4, we show which B&G functions satisfy the properties, and furthermore, we provide examples of non B&G functions that satisfy the properties as well.

4.1 Local-Nash and global-Nash networks

Let a network be called *proper* if the outdegree of each agent is at most one. An illustrative proper network is depicted in Figure 3.

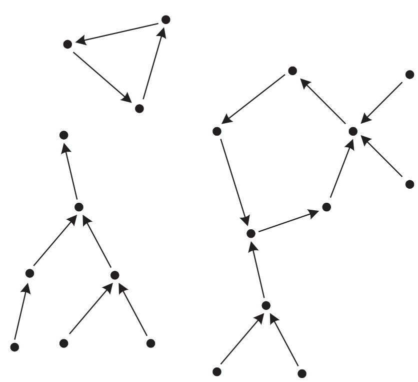

Figure 3: A proper network

Further, let agent i be a *topagent* in network g whenever he observes all agents in his component, i.e. $N_i(g^i) \setminus \{i\} = \mathrm{Car}(g^i)$. Notice that a topagent either is contained in a directed cycle, or he has no outgoing links. Further, notice that in proper networks also the converse holds.

We show that for a specific class of payoff functions every proper local-Nash network is also global-Nash. This class consists of all payoff functions that have three properties which we will define next: **DA** (short for disjoint additivity), **NA** (short for naturality), and **DE** (short for downstream efficiency).

Two networks g and g' are said to be *disjoint* with respect to an agent i, or i-disjoint, if no agent or only agent i is active in both g and g': $\mathrm{Car}(g) \cap \mathrm{Car}(g') \subseteq \{i\}$.

Property DA We say that a payoff function π is *disjoint additive* (**DA** for short), if for each two networks g and g', disjoint w.r.t. an agent i, we have

$$\pi_i(g + g') = \pi_i(g) + \pi_i(g').$$

Let a link (j, i) be called *profitable* if:

$$\pi_i(g) \geq \pi_i(g - (j, i)) \quad \text{if } (j, i) \in g,$$

and

$$\pi_i(g + (j, i)) \geq \pi_i(g) \quad \text{if } (j, i) \notin g.$$

Thus, each profitable link that is not present yet, is worth adding, and each non-profitable link that is present, is worth deleting. Although the notion of profitability is very intuitive in network formation, we propose another notion that indicates the importance of a link.

Let a link (j, i) be *beneficial* in g if it is profitable in g^j_{-i}, i.e.

$$\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i}).$$

Observe that (j, i) does not have to be contained in g. A network is called *beneficial* if the existing links in that network are beneficial.

Lemma 4 *If network g is proper, and π is disjoint additive, then profitability and beneficiality are equivalent notions for existing links; more specifically,*

$$\pi_i(g) - \pi_i(g - (j, i)) = \pi_i(g_{-ij}) - \pi_i(g^j_{-i}) \qquad \textit{for all } (j, i) \in g \tag{2}$$

$$\pi_i(g + (j, i)) - \pi_i(g) = \pi_i(g_{-ij}) - \pi_i(g^j_{-i}) \qquad \textit{for all } (j, i) \notin g \textit{ with } j \notin N_i(g) \tag{3}$$

Proof. First, let $(j, i) \in g$. Since g is proper, g_{-ij} and $g - g_{-ij}$ are i-disjoint. Therefore, by **DA** we have

$$\pi_i(g) = \pi_i(g_{-ij}) + \pi_i(g - g_{-ij}). \tag{4}$$

Also, g^j_{-i} and $g - g^j_{-i}$ are i-disjoint. Hence by **DA** we obtain

$$\pi_i(g - (j, i)) = \pi_i(g^j_{-i}) \ + \pi_i(g - g^j_{-i} - (j, i)). \tag{5}$$

With $g - g_{-ij} = g - g^j_{-i} - (j, i)$, we obtain (2) from (4) and (5).

Now, let $(j, i) \notin g$, with $j \notin N_i(g)$. Networks g_{-ij} and $g - g^j_{-i}$ are i-disjoint, because suppose otherwise: then, an agent $k \neq i$ exists, active in both g_{-ij} and $g - g^j_{-i}$. This implies that $(k, i) \in g$, because any other link attached to k is inside g_{-ij}, and therefore not present in

$g - g^j_{-i}$. Since g is proper, and k has one outgoing link in it (link (k, i)), he cannot have other outgoing links. Therefore, k is a topagent in g^j_{-i}, and thus observes all agents who are active in g^j_{-i}, including j. This implies that $j \in N_i(g)$, which is a contradiction. By **DA** we obtain

$$\pi_i(g + (j, i)) = \pi_i(g_{-ij}) + \pi_i(g - g^j_{-i}). \tag{6}$$

Since g_{-ij} and $g - g^j_{-i}$ are i-disjoint, networks g^j_{-i} and $g - g^j_{-i}$ are also i-disjoint. By **DA** we obtain

$$\pi_i(g) = \pi_i(g^j_{-i}) \; + \pi_i(g - g^j_{-i}). \tag{7}$$

Hence, (3) follows from (6) and (7). □

The next payoff property states that connecting to an agent who is already observed is not a strictly improving action.

Property NA We say that π is *natural* (**NA** for short) if

$$\pi_i(g + (k, i)) \leq \pi_i(g)$$

whenever $k \in N_i(g)$, i.e. there is a directed path from k to i in the network g.

Thus, in a network where i already observes k via another link, say (j, i), the addition of (k, i) is not an improving action due to **NA**. The next payoff property can be seen as a "twin" property.

Property DE Payoff function π satisfies **DE** (short for *downstream efficiency*) if

$$\pi_i(g + (k, i)) \leq \pi_i(g + (j, i))$$

for any network g where $(j, i) \notin g$ and $(k, i) \notin g$ and where a directed path exists from k to j in g_{-i}.

Due to **DE**, the addition of link (j, i) is at least as good as the addition of (k, i). Observe that the difference between **NA** and **DE** is that in the situation where **NA** is applicable, link (j, i) does exist (on the directed path from k to i), whereas in the situation where **DE** is applicable, link (j, i) does not exist.

In the following theorem we show that proper local-Nash networks are also global-Nash whenever the payoff function satisfies the three introduced properties.

Theorem 5 *Let the payoff function π satisfy **DA**, **NA** and **DE**. Then each proper local-Nash network is global-Nash.*

Proof. Let g be a proper local-Nash networks. Suppose to the contrary that g is not global-Nash, say i can strictly improve in g. Let $S = N^d_i(g)$ be his current action, and let $\tilde{S}$ be a strictly improving action, such that $|\tilde{S} \setminus S|$ is as small as possible and such that among those, $|S \setminus \tilde{S}|$ is as small as possible. Let $\tilde{g}$ be the network obtained after i plays $\tilde{S}$.

Let $j \in \tilde{S} \setminus S$.

Suppose that $j \in N_i(g)$. Then agent i observes j via, say, agent $k \in S$ in g, i.e. there is a directed path from j to k in $g_{-i} = \tilde{g}_{-i}$. By **DE** the action $\tilde{S} - j + k$ is at least as good as $\tilde{S}$. Now $|(\tilde{S} - j + k) \setminus S| < |\tilde{S} \setminus S|$, so that we have a contradiction.

Hence $j \notin N_i(g)$. Suppose that networks $\tilde{g} - g_{-ij}$ and g_{-ij} are not i-disjoint. Then an agent $k \in \tilde{S} \cap \mathrm{Car}(g^j_{-i})$ exists. By **DE** we may assume that j is a topagent in g_{-i}, and therefore a directed path exists from k to j in g_{-i}. Hence, by **NA**, it follows that $\tilde{S} - k$ is at least as good as $\tilde{S}$. Now $|(\tilde{S} - k) \setminus S| < |\tilde{S} \setminus S|$; a contradiction. Therefore, networks $\tilde{g} - g_{-ij}$ and g_{-ij} are i-disjoint. Also, $\tilde{g} - g_{-ij} = \tilde{g} - (j,i) - g^j_{-i}$ and g^j_{-i} are i-disjoint.

By **DA** we obtain

$$\pi_i(\tilde{g}) = \pi_i(\tilde{g} - g_{-ij}) + \pi_i(g_{-ij}), \quad \text{and} \tag{8}$$

$$\pi_i(\tilde{g} - (j,i)) = \pi_i(\tilde{g} - (j,i) - g^j_{-i}) + \pi_i(g^j_{-i}). \tag{9}$$

Since g is local-Nash, we have $\pi_i(g + (j,i)) \leq \pi_i(g)$. By Lemma 4, it follows that $\pi_i(g_{-ij}) \leq \pi_i(g^j_{-i})$. Hence by (8) and (9) we obtain $\pi_i(\tilde{g}) \leq \pi_i(\tilde{g} - (j,i))$. Hence $\tilde{S} - j$ is at least as good as $\tilde{S}$, with $|(\tilde{S} - j) \setminus S| < |\tilde{S} \setminus S|$. This is a contradiction.

We conclude that $\tilde{S} \subseteq S$.

Let $j \in S \setminus \tilde{S}$.

Suppose that $j \in N_i(\tilde{g})$, say $(j,k) \in \tilde{g}$. Then, also $(j,k) \in g$. Since $(j,i) \in g$, agent j has two outgoing links, which is a contradiction with the properness of g. We conclude that $j \notin N_i(\tilde{g})$.

Since g is proper local-Nash, (j,i) is profitable in g and by Lemma 4 also beneficial in g. Since $g^j_{-i} = \tilde{g}^j_{-i}$, link (j,i) is also beneficial in $\tilde{g}$. Further, since $\tilde{g} \subset g$, network $\tilde{g}$ is also proper. By Lemma 4, (j,i) is also profitable in $\tilde{g}$. Hence $\tilde{S} + j$ is at least as good as $\tilde{S}$, with $|S \setminus (\tilde{S} + j)| < |S \setminus \tilde{S}|$. This is a contradiction.

Hence we conclude that $\tilde{S} = S$, which contradicts $\tilde{S}$ being a strict improvement. Therefore, g is global-Nash. □

For more general payoff functions, proper local-Nash networks need not to be global Nash as can be seen by the following example.

Example 6 *Consider the following payoff function:*

$$\pi_i(g) = \left|\left\{j \in N^d_i(g) : \; j \notin N_i(g - (j,i))\right\}\right|. \tag{10}$$

*This payoff function satisfies **DA**, since it can be written as the sum over payoff's w.r.t. i-disjoint subnetworks. Further, **NA** is satisfied because the addition of link (k, i) to a network in which a directed path from k to i already exists is not profitable. However, it does not satisfy **DE**, as the following instance shows.*

Consider a network on 4 agents. Let the payoff function $\pi_1(g)$ be defined as (10), and let $\pi_2(g) = \pi_3(g) = \pi_4(g) = 0$ for all networks g. Then, the network depicted in Figure 4(a) is a proper local-Nash network. However, it is not global-Nash, because agent 1 can switch to the network depicted in Figure 4(b), which yields payoff 2 instead of 1. This network is global-Nash.

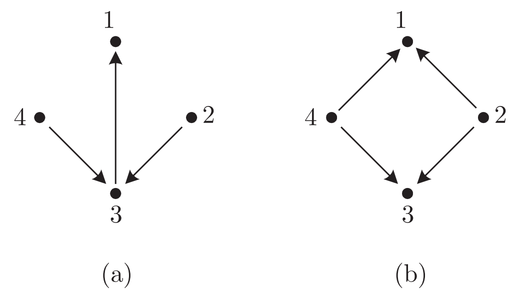

Figure 4: A proper local-Nash network that is not global-Nash (a), and a global-Nash network (b).

4.2 Existence of local-Nash networks

Next, we identify a class of properties for which we prove the existence of proper local-Nash networks. For this, we introduce four new properties. These properties only regard beneficiality. Recall that a link (j, i) is beneficial in g whenever $\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i})$. The properties concern how beneficiality of a link is preserved when the network is changed, or how beneficiality of a link depends on the beneficiality of other links.

Property BT Payoff function π satisfies **BT** (short for *beneficial topagent*) if the following holds. Let link (k, i) be beneficial in network g, and suppose there are topagents in the component g^k_{-i}. Then there is a topagent j in g^k_{-i} such that $\pi_i(g_{-ij}) \geq \pi_i(g_{-ik})$.

Notice that this property is implied by **DE**. The following property is also implied by **DE**.

Property BF Payoff function π satisfies **BF** (short for *beneficial farthest*) if the following holds. Let link (k, i) be beneficial in network g; let the component g^k_{-i} be proper and let agent i be active in g^k_{-i} (there is an outgoing link at i in g^k_{-i}). Then also link (j, i) is beneficial where j is the agent farthest away from i (counted in number of links) in network g.

Notice that since component g^k_{-i} is proper and i is active in it, agent j is the unique topagent who is farthest away from i in network g. Property **BF** is also implied by **DE**. However, it is independent of **BT** as we will see in subsection 4.3. Furthermore, **BT** and **BF** do not imply **DE**. This is illustrated by Example 6, where the payoff function given by (10) does not satisfy **DE**, whereas it satisfies both **BT** and **BF**, since $\pi_i(g_{-ij}) = 1$ and $\pi_i(g^j_{-i}) = 0$ for each network g and each agent j.

The following property describes that beneficial links remain beneficial while the network grows:

Property BG Payoff function π satisfies **BG** (short for *beneficial growth*) if $\pi_i((g+(k,r))_{-ij}) \geq \pi_i((g+(k,r))^j_{-i})$ for any two agents k, r, whenever $\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i})$.

Notice that in case we have $r = i$, **BG** trivially holds, since $(g + (k, r))_{-ij} = g_{-ij}$.

The final property states that beneficiality is preserved when we delete a spoke from the network. Here, a *spoke* in a network g is a link (k, r) such that both agents k and r reside on a directed cycle, with link (k, r) not being part of it.

Property BS Payoff function π satisfies **BS** (*beneficial shrink*) if $\pi_i((g-(k,r))_{-ij}) \geq \pi_i((g+(k,r))^j_{-i})$ whenever $\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i})$ and link (k,r) is a spoke in g.

The properties **BF**, **BG** and **BS** are trivially satisfied by payoff functions for which $\pi_i(g) \geq \pi_i(g_{-i})$ for all networks g. An example of such a function is $\pi_i(g) = |N_i(g) \setminus \{i\}|$ being the number of agents in g observed by i. This function also satisfies **DA**, **NA** and **BT**.

Let us call a payoff function *orderly* if it satisfies the properties **DA**, **NA**, **BT**, **BF**, **BG**, and **BS**.

Let $\kappa(g)$ be the *connection number* of network g, defined as

$$\kappa(g) = \sum_{i \in N} |N_i(g)|.$$

Observe that the addition of a link or the deletion of a spoke does not decrease the connection number. Now, we state our main result:

Theorem 7 *For orderly payoff functions any proper, beneficial network with maximal connection number is a local-Nash network.*

Proof. Observe that the empty network is proper and beneficial. So, there is a proper, beneficial network, say g, such that among these networks the connection number $\kappa(g)$ is maximal. We prove that g is local-Nash by deriving a contradiction in the sense that otherwise a proper, beneficial network exists with a higher κ-value than g.

Suppose there is a local action by agent i that strictly increases i's payoff. Clearly, this action is not a pass. This local action is neither a deletion because all links of agent i are beneficial, and therefore also profitable by Lemma 4 due to g being proper and π_i satisfying **DA**.

Suppose the strictly improving local action is a replacement, say link (k,i) is replaced by link (j,i), and let the obtained network be $\tilde{g} = g-(k,i)+(j,i)$. Notice that k is the unique topagent in g^k_{-i}, since g is proper. If both agents k and j are in component g^k_{-i}, then, by property **BT** it follows that $\pi_i(g_{-ij}) \leq \pi_i(g_{-ik})$. However, $g - g_{-ik}$ and $\tilde{g}_{-ij} = g_{-ij}$ are i-disjoint and their union is $\tilde{g}$, so that by **DA** we have

$$\pi_i(g - g_{-ik}) + \pi_i(g_{-ik}) = \pi_i(g) < \pi_i(\tilde{g}) = \pi_i(g - g_{-ik}) + \pi_i(g_{-ij}),$$

i.e., $\pi_i(g_{-ij}) > \pi_i(g_{-ik})$; a contradiction.

Therefore, agents k and j are in different components of g_{-i}. The networks g_{-ik} and $g - g_{-ik} + (j,i)$ are i-disjoint, with union equal to $g+(j,i)$, and the networks g^k_{-i} and $g-g_{-ik}+(j,i)$ are i-disjoint, with union $g-(k,i)+(j,i)$. Applying **DA** twice, we obtain

$$\begin{aligned}
\pi_i(g+(j,i)) &= \pi_i(g - g_{-ik} + (j,i)) + \pi_i(g_{-ik}) \\
&\geq \pi_i(g - g_{-ik} + (j,i)) + \pi_i(g^k_{-i}) \\
&= \pi_i(g - (k,i) + (j,i)).
\end{aligned}$$

The inequality follows from the beneficiality of (k,i). We conclude that the addition of (j,i) is at least as good as the replacement of link (k,i) in g by (j,i).

So, we may assume that the strict improving local action is an addition. Let this addition be (j, i) and let the obtained network be

$$g' = g + (j, i). \tag{11}$$

If the component $(g')^j_{-i}$, which is equal to g^j_{-i}, is already linked up with i, say $(k, i) \in g$ and $k \in \text{Car}(g^j_{-i})$, then k is the unique topagent in g^j_{-i}, due to the properness of g. So, there is a directed path from j to k in g, and with (k, i) there is a directed path from j to i in g, implying $\pi_i(g') = \pi_i(g + (j, i)) \leq \pi_i(g)$ because of **NA**. This is a contradiction to the fact that adding (j, i) is strictly improving.

Therefore, the component $(g')^j_{-i} = g^j_{-i}$ is not linked up with i in g, and by **BT** we may assume that j is a topagent in g^j_{-i}.

Since g is proper, and since $j \notin N_i(g)$, by Lemma 4 and **DA**, link (j, i) is beneficial in g'. Also, the other links are beneficial in g' due to the beneficiality of g and **BG**. So, g' is a beneficial network. Further, the number $\kappa(g')$ is higher than $\kappa(g)$, so that g' cannot be proper, because by assumption there cannot be proper and beneficial networks with connection number higher than $\kappa(g)$. The only outdegree changed by going from g to g' is the one of agent j. Therefore, the outdegree of j in g' equals 2, say next to link (j, i), also (j, k) is present in g'. Since j is a topagent in g^j_{-i} there is a directed path from k to j in g^j_{-i}. Observe that this is also a directed path in $(g')^j_{-k}$.

Extending the directed path from k to j via (j, i) in $(g')^j_{-k}$ in a unique way (since it is a proper network), we arrive at an agent, say r, farthest away from k (see Figure 5). Since (j, k) is beneficial for k, in g', also (r, k) is beneficial in g' due to **BF**.

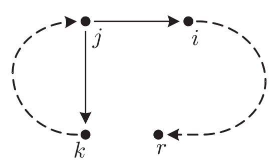

Figure 5: The addressed links and directed paths (dashed arcs) of network g'.

Consider the addition of (r, k) in g'. From **BG** and (r, k) being beneficial in g', we conclude that $g' + (r, k)$ is beneficial. Further, (j, k) is a spoke in this network. After deletion of this spoke, by **BS** we again obtain a beneficial network

$$g'' = g' + (r, k) - (j, k), \tag{12}$$

with a connection number at least as high as $\kappa(g')$ and thus higher than $\kappa(g)$. Hence g'' cannot be proper. This implies that the outdegree of agent r is greater than 1 in g''. Besides (r, k) we have another link, say (r, s), and s is necessarily located on the unique directed path from k to r in g', for otherwise, s would be farther away from k than r is.

This directed path also exists in g'', and together with (r, k) it forms a directed cycle in g'' with (r, s) being a spoke of it (see Figure 6). By deletion of (r, s) we obtain a beneficial network

$$g''' = g'' - (r, s), \tag{13}$$

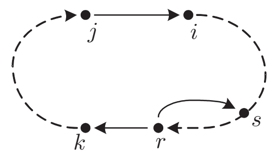

Figure 6: Network g'', where s is located on the directed path from k to r.

due to **BS**. Its connection number is higher than the one of g. Observe that g''' is proper: agents j and r (the only agents whose outgoing links are changed w.r.t. g) both have exactly one outgoing link in g'''. This is a contradiction by our assumption that g is a proper network with maximal connection number. We conclude that there are no strictly improving additions available, i.e., g is local-Nash. □

Proper, beneficial networks with maximal connection numbers are not the only local-Nash networks. The following example shows that even among the non-proper networks local-Nash networks may be found.

Example 8 *Consider the B&G function $\pi_i(g) = |N_i(g)| - |N_i^d(g)|$. This payoff function is orderly as we will see in subsection 4.4. The network depicted in Figure 7 is local-Nash, but not proper since agent i has two outgoing links.*

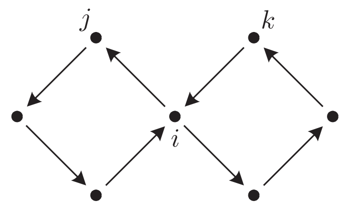

Figure 7: A local-Nash network that is not proper.

Notice that this network is not strict local-Nash, because the replacement of (j, i) by (j, k) yields the same payoff for agent j. When agent i subsequently removes the spoke (k, i), we obtain a local-Nash network, which is also proper.

The following example shows that even *strict* local-Nash networks may not be proper.

Example 9 *For even n and $n \geq 4$, let $\tilde{g}$ be the following network architecture. Let all agents be contained on one undirected cycle, where the directions of the links are alternated. In Figure 8, network $\tilde{g}$ is depicted for $n = 14$.*

Let π be a payoff function, for each agent i defined as $\pi_i(g) = 1$ whenever $g = \tilde{g}$, and $\pi_i(g) = 0$ otherwise. Since network $\tilde{g}$ is the unique network for which each agent yields a payoff strictly higher than 0, it is a strict local-Nash network.

Now we show that π is orderly.

***DA** is satisfied, since $\tilde{g}$ cannot be the union of two i-disjoint networks, and it is not disjoint from any non-empty network.*

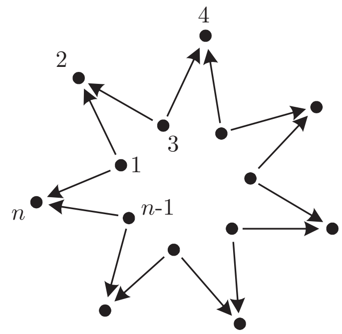

Figure 8: Example of network $\tilde{g}$

*Property **NA** is also satisfied, because of the following. For any network g where a directed path of at least two links exists from k to i, it follows that $\pi_i(g + (k, i)) = \pi_i(g) = 0$.*

*Since $\pi_i(g_{-ij}) = \pi_i(g_{-i}) = 0$ for all networks g, all links are beneficial in any network. Therefore, **BF**, **BS**, and **BG** are also satisfied. Further, since all links are equally beneficial, property **BT** is also satisfied.*

When we relate the previous theorem with Theorem 5 (proper local-Nash networks are global-Nash if the payoff function satisfies **DA**, **NA** and **DE**) and with the observation that **BT** and **BF** are implied by **DE**, we obtain the following corollary.

Corollary 10 *For any payoff function that satisfies **DA**, **NA**, **DE**, **BG** and **BS**, global-Nash networks exist. Specifically, the proper and beneficial networks with maximal connection number are global-Nash.*

4.3 Property independence

In this subsection, we show the independence of the six properties that define orderliness. This is done by an exposition of examples of payoff functions, fulfilling all but one property.

Theorem 11 *The properties **DA**, **NA**, **BT**, **BF**, **BG**, **BS** are independent of each other.*

Proof. We show that for each property a payoff function exists which does not satisfy that property while it does satisfy all other properties.

(all but **DA**) The following payoff function satisfies all properties, except **DA**:

$$\pi_i(g) = |N_i(g) \setminus \{i\}|^2 \tag{14}$$

Property **DA** is not satisfied, because for any two i-disjoint networks g and g' we have $|N_i(g)\setminus\{i\}|^2 + |N_i(g')\setminus\{i\}|^2 < |N_i(g \cup g')\setminus\{i\}|^2$. The properties **NA** and **BT** are trivially satisfied and the others because $\pi_i(g) \geq \pi_i(g^j_{-i})$ for all networks g.

(all but **NA**) The following payoff function satisfies all properties, except **NA**:

$$\pi_i(g) = |N_i^d(g)| \tag{15}$$

Property **NA** is not satisfied, because $\pi_i(g+(k,i)) > \pi_i(g)$ for a network g where $(k,i) \notin g$, and where a directed path from k to i exists. Property **DA** is clearly satisfied, and also the four properties that concern beneficiality, because $\pi_i(g_{-ij}) = 1$ for all g and j.

(all but **BT**) Let agent 1 be a special member of N, and let $i \in N \setminus \{1\}$. The following payoff function satisfies all properties, except **BT**:

$$\pi_i(g) = \begin{cases} 1 & \text{if } 1 \in N_i^d(g) \text{ and } 1 \notin N_i(g-(1,i)); \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

This payoff function does not satisfy **BT**, because in any network g^1_{-i} where agent 1 is not a topagent, we have $\pi_i(g_{-i1}) = 1$, while $\pi_i(g_{-ij}) = 0$ for each topagent $j \in \text{Car}(g^1_{-i})$. It can be easily verified that properties **NA** and **DA** are satisfied. The remaining properties are also satisfied since $\pi_i(g^j_{-i}) = 0$ and $\pi_i(g_{-ij}) \geq 0$ for each network g and each agent j.

(all but **BF**) Let agent 1 be a special member of N, and let $i \in N \setminus \{1\}$. The following payoff function satisfies all properties, except **BF**.

$$\pi_i(g) = \begin{cases} 0 & \text{if } N_i(g) = N, 1 \notin N_i^d(g), |N_i^d(g)| = 1; \\ -|N_i^d(g)| & \text{otherwise.} \end{cases} \tag{17}$$

Observe that $\pi_i(g_{-i}) = 0$ for all networks g.

Property **BF** is not satisfied, because of the following. Let g be a network where all agents in $N \setminus \{i\}$ are contained in one directed cycle, and let link (i,k) also be present. Further, let 1 be the agent who is farthest away from i (so, $(1,k) \in g$). Then $\pi_i(g_{-ik}) = 0$ while $\pi_i(g_{-i1}) = -1$.

It can be easily but tediously verified that property **DA** is satisfied. Property **NA** is satisfied, because adding a link (k,i) to network g where a directed path from k to i exists, implies that i will have multiple links, and therefore his payoff will decrease. Property **BT** is satisfied because only link $(1,i)$ can be beneficial, which is the case only if $N_i(g_{-i1}) = N$, which implies that 1 is a topagent. By similar reasoning, properties **BG** and **BS** are satisfied.

(all but **BG**) The following payoff function satisfies all properties, except **BG**.

$$\pi_i(g) = \left|N_i^d(g) \cap T(g_{-i})\right| - \left|N_i^d(g) \setminus T(g_{-i})\right| \tag{18}$$

where $T(g_{-i})$ is the set of topagents in g_{-i} who do not have outgoing links.

This payoff function does not satisfy **BG**, because due to the addition of link (k, r), $r \neq i$ to g, we may have that $|T((g + (k, r))_{-i})| < |T(g_{-i})|$, such that $\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i})$ and $\pi_i((g + (k, r))_{-ij}) < \pi_i((g + (k, r))^j_i)$. Property **DA** is satisfied, because the sets $N^d_i(g)$ and $T(g)$ can be decomposed into disjoint subsets w.r.t. i-disjoint subnetworks. Property **NA** is satisfied, because in a network g where a directed path exists from k to i, we have $k \notin T(g)$, and therefore the payoff does not increase when (k, i) is added. Property **BT** is trivially satisfied since (k, i) can only be beneficial if k is a topagent in g_{-i}, by (18). Furthermore, a topagent k in g_{-i} such that (k, i) is beneficial, has no outgoing links in g_{-i} (since $k \in T(g_{-i})$). Therefore, if g is proper, and if i has an outgoing link to the component g^k_{-i}, then it follows that k is the agent who is farthest away from i in g. Hence **BF** is also satisfied. Property **BS** is satisfied because the deletion of a spoke (k, r) in g does neither affect the set $T(g_{-i})$ nor the set $N^d_i(g)$.

(all but **BS**) The following payoff function satisfies all properties, except **BS**.

$$\pi_i(g) = \left|K_i(g)\right| - \left|N^d_i(g)\right| \tag{19}$$

where $K_i(g)$ is the set of spokes that i views in g, i.e.

$$K_i(g) = \{(k, r) : \ r \in N_i(g) \text{ and } (k, r) \text{ is a spoke}\}$$

Property **BS** is not satisfied, because by removing a spoke (k, r) in a network g, the cardinality of $K_i(g)$ may decrease such that $\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i})$ and $\pi_i((g - (k, r))_{-ij}) < \pi_i((g - (k, r))^j_{-i})$. Clearly, this payoff function satisfies **NA** and **DA**. For the properties **BT** and **BF** and **BG** notice that the payoff $\pi_i(g_{-ij})$ only depends on the number of spokes viewed in g_{-ij}. Properties **BT** and **BF** are satisfied, because of the following. Let k be an agent in a network g and let j be a topagent in g^k_{-i}. Since i views at least as many spokes in g_{-ij} as in g_{-ik}, **BT** and **BF** are satisfied. Property **BG** is satisfied because for any network g and any agent j, the number $\left|K_i(g_{-ij})\right|$ cannot decrease by adding a link to g.

□

We already observed that **DE** implies **BT** and **BF**, and not vice versa. In the following theorem, we show that the properties that are needed for Corollary 10, which are **NA**, **DA**, **DE**, **BS** and **BG**, are independent of each other as well.

Theorem 12 *The properties **DA**, **NA**, **DE**, **BG**, **BS** are independent of each other.*

Proof. By Theorem 11 we know that **DA**, **NA**, **BG** and **BS** are independent of each other. Therefore it remains to show that **DA**, **NA**, **BG** and **BS** are independent of **DE**.

Payoff function (10) in Example 6 does not satisfy **DE**, whereas it satisfies all other properties. In the example it was shown that **DA** and **NA** are satisfied. The other properties, **BG** and **BS**, are trivially satisfied since $\pi_i(g^j_{-i}) = 0$ and $\pi_i(g_{-ij}) = 1$ for all g and j. Hence **NA**, **DA**, **BG** and **BS** do not imply **DE**.

To show that **DE** does not imply **DA**, **NA**, **BG** nor **BS**, consider the payoff functions (14), (15), (18) and (19). They do not satisfy **DA**, **NA**, **BG** and **BS** respectively. However, it can be easily checked that these functions do satisfy **DE**.

We conclude that the properties **DA**, **NA**, **DE**, **BG**, **BS** are independent of each other. □

4.4 Relationship with B&G payoff functions

In this subsection, we analyze B&G functions in view of the framework of payoff properties as discussed previously. We prove that B&G functions with owner-homogeneous link costs and heterogeneous profits are orderly and also satisfy **DE**. Then, we prove that B&G functions with heterogeneous link costs that satisfy a system of triangle inequalities, are orderly without necessarily satisfying **DE**. Further, we provide several examples of payoff functions that satisfy all properties, while they fall outside the class of B&G functions.

For B&G functions, we may assume that $v_{ii} = 0$, because the transformation $\pi'_i(g) = \pi_i(g) - \pi_i(g_{-i}) = \pi_i(g) - v_{ii}$ does not have influence on the strategic behavior of agent i.

In the next lemma, we prove that all B&G functions satisfy four properties.

Lemma 13 *Let π_i be a B&G function. Then π_i satisfies* ***DA**, **NA**, **BG** and **BS**.*

Proof.
(**DA**) For each two i-disjoint networks g and g' it holds that $N_i(g) \cap N_i(g') = \{i\}$ and $N_i^d(g) \cap N_i^d(g') = \emptyset$. Since we assumed that $v_{ii} = 0$, it follows that $\pi_i(g+g') = \pi_i(g) + \pi_i(g')$. Therefore π satisfies **DA**.

(**NA**) If a directed path exists from k to i in network g where link (k,i) does not exist, then $N_i(g) = N_i(g + (k,i))$, and $N_i^d(g) \subset N_i^d(g + (k,i))$. Hence property **NA** is satisfied.

(**BG**) Let g be a network where (j,i) is beneficial. Since $N_i(g_{-ij}) \subseteq N_i((g + (k,r))_{-ij})$ and $N_i^d(g_{-ij}) = N_i^d((g + (k,r))_{-ij}) = 1$, property **BG** is satisfied.

(**BS**) Let g be a network that contains a spoke (k,r). Let (j,i) be beneficial in g. Since $N_i(g_{-ij}) = N_i((g - (k,r))_{-ij})$ and $N_i^d(g_{-ij}) = N_i^d((g - (k,r))_{-ij}) = 1$, link (j,i) is also beneficial in $g - (k,r)$. Hence **BS** is satisfied. □

In the next result, we show that B&G functions with owner-homogeneous link costs satisfy all properties, and therefore imply the existence of global-Nash networks (by Corollary 10). This result is also proved by Billand et al. (2008) and independently by Derks et al. (2008).

Theorem 14 *Let π be a B&G function with owner-homogeneous link costs, i.e. $c_{ij} = c_i$ for all $i,j \in N$. Then π satisfies* ***DA**, **NA**, **DE**, **BG**, and **BS**, i.e. global-Nash networks exist.*

Proof. By Lemma 13 it follows that π satisfies **DA**, **NA**, **BG** and **BS**.

Let g be a network where $(j,i) \notin g$, $(k,i) \notin g$, and where a directed path exists from k to j in g_{-i}. Then $N_i(g + (j,i)) \supseteq N_i(g + (k,i))$ and $|N_i^d(g + (j,i))| = |N_i^d(g + (k,i))|$. Hence, property **DE** is satisfied. □

Global-Nash networks do not exist for B&G functions with heterogeneous link costs in general. Even if these B&G functions are restricted by specific conditions, the existence of global-Nash networks is not guaranteed. This is illustrated by an example provided by Derks et al. (2008). In this example, global-Nash networks do not exist, while the link costs are arbitrarily close to the situation of owner-homogeneity, i.e. $|c_{ij} - c_{ik}| \leq \epsilon$, for all $i,j,k \in N$ and an arbitrarily $\epsilon > 0$.

The existence of local-Nash networks is proved in Theorem 7 for orderly payoff functions. Notice that these payoff functions satisfy **BT** and **BF** instead of **DE** (which implies both of them). In the next theorem we provide conditions for B&G function with heterogeneous link costs such that these functions are orderly.

Theorem 15 *Let π_i be a B&G function with heterogeneous link costs and profits. If*

$$c_{ij} \leq v_{ij} + min(v_{ik}, c_{ik}), \text{ for all } j, k \in N, \tag{20}$$

then π_i is orderly, i.e. local-Nash networks exist.

Proof. By Lemma 13, the properties **DA**, **NA**, **BG** and **BS** are satisfied. It remains to prove that π satisfies **BT** and **BF**:
(**BT**) Let link (k,i) be beneficial in g. Then $\sum_{r \in N_i(g_{-ik})} v_{ir} \geq c_{ik}$. If a topagent j exists in the component g^k_{-i}, then either $k = j$ or a directed path from k to j exists. In the first case **BT** is trivially satisfied. In the second case, it follows that $N_i(g_{-ij}) \supseteq N_i(g_{-ik}) \cup \{j\}$. Since $c_{ij} \leq v_{ij} + c_{ik}$ we have

$$\begin{aligned}
\pi_i(g_{-ik}) &= \Big(\sum_{r \in N_i(g_{-ik})} v_{ir}\Big) - c_{ik} \\
&\leq \Big(\sum_{r \in N_i(g_{-ik})} v_{ir}\Big) - (c_{ij} - v_{ij}) \\
&\leq \Big(\sum_{r \in N_i(g_{-ij})} v_{ir}\Big) - c_{ij} \\
&= \pi_i(g_{-ij}).
\end{aligned}$$

Hence **BT** is satisfied by π.
(**BF**) Let g^k_{-i} be a proper component of g where i has an outgoing link and let link (k,i) be beneficial in g. Let j be a topagent in this component who is farthest away from i. If $k = j$ then **BF** is trivially satisfied. Otherwise a path from k to j exists. Therefore both agents j and k are contained in $N_i(g_{-ij})$. Since $c_{ij} \leq v_{ij} + v_{ik}$ it follows that $\pi_i(g_{-ij}) \geq v_{ij} + v_{ik} - c_{ij} \geq 0$. Hence **BF** is satisfied. □

For a full characterization of B&G functions that satisfy the properties of our framework, and hence imply the existence of local-Nash networks, we refer to Derks and Tennekes (2007).

Observe that local- and global-Nash networks also exist for B&G functions under conditions that are weaker than (20):

Proposition 16 *Let π be a B&G function with heterogeneous link costs and profits. If*

$$c_{ij} \leq \begin{cases} v_{ij} + c_{ik} & \text{for all } i, j, k \in N, \\ \sum_{j \in N} v_{ij} \end{cases} \tag{21}$$

then any directed cycle joining all agents is a local- and global-Nash network.

The proof is left to the reader. Also without proof we notice that a B&G function that fulfills (21) does not necessarily satisfy **BF**.

Our framework of properties is also satisfied by non B&G payoff functions. Consider the following examples:

$$\begin{aligned}
\pi_i(g) &= |N^d_i(g) \cap T(g_{-i})|; \\
\pi_i(g) &= |K_i(g)|; \\
\pi_i(g) &= |C(g) \cap N_i(g)| - |N^d_i(g)|,
\end{aligned}$$

where $C(g)$ is the set of agents that are contained in a directed cycle in g, and where $T(g)$ and $K_i(g)$ are respectively defined as the set of topagents in g who do not have outgoing links, and the set of spokes in g that i observes. These payoff functions are orderly and also satisfy **DE**. Payoff function (10) in Example 6, which is also studied in the proof of Theorem 12, is a non B&G payoff function that is orderly whereas it does not satisfy **DE**.

These payoff functions extend the class of B&G functions in the following way. They do not only consider which agents are (directly) observed, i.e. which agents are contained in the sets $N_i(g)$ and $N_i^d(g)$. They also take other aspects of the network architecture into account. In the given examples, the sets $T(g_{-i})$, $K_i(g)$, and $C(g)$ illustrate this.

5 Dynamics

In this section we analyze a dynamic process of iterated local actions that takes place without central coordination. We consider a procedure in which the agents alternately play good local responses. Recall that an agent plays a good response, if his payoff does not decrease. If this payoff remains the same, then we say that this agent plays a *neutral response*.

The dynamic procedure that we study in this paper, starts with an arbitrary initial network. Then, one agent is selected at random. One of his good local responses is selected at random, and being played. These steps are repeated. Formally, we define the procedure on base of the following assumptions.

A-1 Let the initial network be a network that is arbitrarily chosen from $\mathcal{G}$.

A-2 At the beginnig of each stage, an agent is selected at random, where each agent has a positive stage independent probability to be selected.

A-3 At each stage, the agent who is selected plays a good local response that satisfies the following three assumptions.

A-3a A neutral addition is not allowed.

A-3b A neutral deletion of link (j,i) in network g is only allowed whenever $N_i(g-(j,i)) = N_i(g)$.

A-3c A neutral replacement of (k,i) by (j,i) in network g is only allowed when a directed path exists from k to j in network g_{-i}.

He chooses a good local response at random, where all allowed good local responses have a positive probability to be chosen that only depends on the network.

We say that the procedure *terminates* at a network g if this network is reached, and furthermore, if a pass is the only allowed good local response for each agent i with respect to g.

We need assumptions A-3a to A-3c in order to prevent the following situation. Consider a game where an agent i is present, such that $\pi_i(g) = 0$ for all networks g. When the procedure reaches some local-Nash network, this agent may perform randomly chosen neutral local responses, such that the obtained network is not local-Nash again.

In the following lemma, we show that the procedure defined by A-1 to A-3c terminates whenever a proper local-Nash network is reached.

Lemma 17 *Let the dynamic procedure be defined by assumptions A-1 to A-3c. If this procedure reaches a proper local-Nash network, then it also terminates at this network.*

Proof. Let g be a proper local-Nash network that is reached by the dynamic procedure. Since g is local-Nash, it can only be modified by neutral responses. Let i be an agent who can apply a neutral response to g. We know by assumption A-3a that this action cannot be a neutral addition.

Suppose that this action is a deletion. Since g is proper, each deletion strictly reduces the set of observed agents. By assumption A-3b, these deletions are not allowed. Hence we conclude that this action cannot be a deletion.

Suppose that this action is a replacement. By assumption A-3c, a neutral replacement of (k, i) by (j, i) is only allowed when a directed path exists from k to j that does not visit agent i. In that case, agent k has two outgoing links: (k, i) and a link on the path from k to j. This contradicts that g is proper.

Hence we conclude that the only neutral response that can be applied to g is a pass. Therefore, the procedure terminates at network g. □

We prove that the procedure reaches a proper local-Nash network with probability 1. First, we show that a finite sequence of good local responses exists that can be applied iteratively to any arbitrary network such that the obtained network is local-Nash. This sequence starts with actions such that the initial network is reshaped to a proper and beneficial network. From there, we re-use the result of the proof of Theorem 7 which states that if a proper and beneficial network is not local-Nash, then another proper and beneficial network exists with a higher connection number. Iteratively using this result, we obtain a network with a maximal connection number, which implies that this network is proper, local-Nash.

Lemma 18 *Let π satisfy **DA**, **NA**, and **DE**. Then, for any network in $\mathcal{G}$, there exists a finite sequence of good local responses that leads to a proper and beneficial network.*

Proof.

Step 1 Let $g \in \mathcal{G}$. First we make g proper by applying good local responses. If g is already proper, then continue to step 2. Otherwise an agent i exists in g who has at least two outgoing links, say (i, j) and (i, k) (see Figure 9).

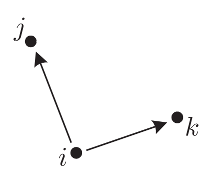

Figure 9: An agent with two outgoing links

Two cases are distinguished:

A: There is a directed path from i to an agent ℓ with outdegree 0, starting with link (i, k). The property **DE** implies that the link (i, j) may be replaced by (ℓ, j). This action decreases the total outdegree of the agents with multiple outgoing links.

B: None of the directed paths starting with link (i,k) end at an agent with outdegree 0. Either there is a directed cycle C containing (i,k), or there is a directed path starting with link (i,k) and ending at an agent ℓ on a directed cycle. In the latter case we may apply the property **DE** and replace link (i,j) with (ℓ,j). It is therefore no loss of generalization to assume a directed cycle with (i,k) in it.

We distinguish four subcases:

1: Agent j is on cycle C. Then a directed path exists from i to j and hence the link (i,j) can be deleted by **NA**.

2: There is a directed path from i to an agent with outdegree 0, and starting with link (i,j). Case **A** addresses this situation.

3: There is a directed cycle C' containing (i,j). Going in the opposite direction over C', let ℓ be the last agent on this cycle who is also on the cycle C through (i,k) (see Figure 10). Using property **DE** we may replace link (i,k) with link (ℓ,k), so that we

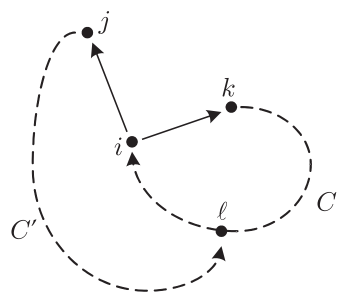

Figure 10: Situation of case 3

can assume that both cycles C and C', have only agent i in common. This situation is depicted in Figure 11.

Let agent ℓ be such that (ℓ,i) is on cycle C (ℓ may be the agent k). Now, replace

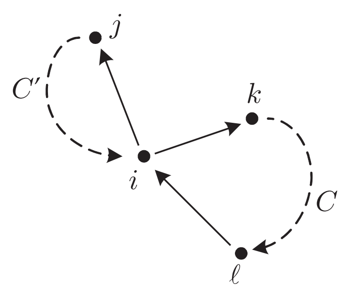

Figure 11: Situation of case 3 continued

(i,j) with (ℓ,j). This is a good local response by **DE** since there is a directed path from i to ℓ, without visiting j. After this replacement, the link (ℓ,i) can be deleted by **NA** since there is a directed cycle in which i,k,ℓ,j are visited in this order, and hence (ℓ,i) is a spoke.

4: there is a directed path starting with link (i, j) and ending at an agent ℓ on a directed cycle. Then agent ℓ is not on the directed cycle through (i, k). (Otherwise, we would have obtained a directed cycle containing (i, j), and this is already taken care of in case 3.) Using property **DE**, we may replace link (i, k) with link (ℓ, j), and by this action we arrive at the situation treated in case 3.

As long as there are agents with outdegree greater than 1, g is not proper, and hence this step can be repeated. Each time it is repeated, the outdegree of one agent is reduced without changing the outdegrees of the other agents. Therefore, after a finite number of repetitions we obtain a proper network.

Step 2 Let g' be the proper network that results from step 1. If g' is not beneficial, then a non-beneficial link (j, i) exists in g'. Since g' is proper and since **DA** is satisfied, by Lemma 4 we know that (j, i) is also not profitable, and therefore the deletion of it is a good local response. Obviously, g' remains proper after this deletion. Such deletions can be applied repeatedly until we obtain a proper and beneficial network g''. □

The next lemma shows that there exists a finite sequence of good local responses that can be applied iteratively to any proper and beneficial network such that it leads to a proper local-Nash network.

From the proof of Theorem 7 we may deduce that from a non local-Nash network which is proper and beneficial, a network can be constructed which is also proper and beneficial but which has a higher κ-value. We show that this construction can be done by a sequence of good local responses.

Lemma 19 *Let π be an orderly payoff function that satisfies **DE**. Let g be a proper and beneficial network. There exists a finite sequence of good local responses that leads to a proper local-Nash network.*

Proof. Suppose that g is not local-Nash. Since g is proper and beneficial, we know from the proof of Theorem 7 that a network can be obtained with a higher connection number. We show that we can obtain this network by applying good local reponses.

Consider the networks g', g'' and g''' as defined in (11), (12), and (13). Network g' is obtained from g by a strictly improving addition, which is trivially a good local response. Network g'' is obtained from g' by a replacement of (j, k) by (r, k) where a directed path from j to r exists in g. By **DE**, this is also a good local response. Finally, network g''' is obtained from g'' by a deletion of spoke (r, s) which is a good local response by **NA**. Observe that g''' is proper and benefical. Therefore, if g''' is not local-Nash, we can repeat these good local responses until we obtain a local-Nash network. At each iteration, the connection number increases. Since this number is bounded by $n(n-1)$, we obtain a local-Nash network in a finite number of iterations. □

Combining Lemma's 18 and 19 we obtain a sequence of networks that starts with an arbitrary initial network and ends with a proper local-Nash network. Observe that the initial network can be a non-proper local-Nash network. In the next theorem we show that our procedure always reaches a local-Nash network.

Theorem 20 *Let π be an orderly payoff function that satisfies **DE**, and let the dynamic procedure be as defined by assumptions A-1 to A-3c. Then this procedure terminates at a proper local-Nash network with probability 1.*

Proof. First we prove that the procedure reaches a local-Nash network with probability 1, and then we prove that it also terminates at this network with probability 1. By Lemma's 18 and 19 we know that from an arbitrary network in $\mathcal{G}$ a finite sequence of good local responses exists, such that the obtained network is proper and local-Nash. It is easily verified that these good local responses satisfy assumptions A-3a to A-3c:

- the only additions in this sequence are strictly improving ones;
- each deletion is either validated as a good local response by **NA** (and hence it satisfies assumption A-3b), or it is a deletion of a non-beneficial link in a proper network which is a strictly improving deletion by **DA**;
- all replacements in this sequence are validated as good local responses by **DE** and hence they satisfy assumption A-3c.

Hence, any sequence that is constructed in the proofs of Lemma's 18 and 19 satisfies the assumptions A-1 to A-3c.

By the construction of such a sequence, we know that each network in $\mathcal{G}$ appears at most once in this sequence. Therefore, we conclude that the length of this sequence is upperbounded by M, the finite number of networks in $\mathcal{G}$.

At any stage, each agent has a strictly positive probability to be chosen (assumption A-2), and each allowed good local response has a strictly positive probability to be chosen (assumption A-3). Therefore, the probability that such a sequence will be played is lowerbounded by a strictly positive probability ϵ.

The probability that the dynamic procedure does not reach a local-Nash network after M steps is lower than $1-\epsilon$. If it does not reach a local-Nash network after M steps, then from the last network, another sequence exists that leads to a local-Nash network. Hence, the probability that the dynamic procedure does not reach a local-Nash network after $2M$ steps is lower than $(1-\epsilon)^2$, and after kM steps lower than $(1-\epsilon)^k$, with k being a strictly positive natural number. Hence we conclude that this probability converges to 0 as k becomes larger. Therefore, this procedure reaches a proper local-Nash network with probability 1.

By Lemma 17 we know that this procedure also terminates at this network with probability 1. □

Combining this result with Theorem 5, saying that each proper local-Nash network is also global-Nash, we obtain the following corollary.

Corollary 21 *Let π be an orderly payoff function that satisfies **DE**, and let the dynamic procedure be as defined by assumptions A-1 to A-3c. Then this procedure terminates at a proper global-Nash network with probability 1.*

6 Conclusion

In this paper, we have studied a dynamic model of unilateral network formation. We have extended the literature on non-cooperative network formation in two ways. First, we introduced a local approach, where agents are restricted to play local actions. Second, we developed a framework of axiomatic payoff properties.

We proved the existence of local-Nash and global-Nash networks for games with payoff functions that satisfy these properties. Further, we proved that our iterative procedure of local actions always terminates at a local-Nash network, which is also global-Nash. In this context, one way to continue the research is to obtain insights in the speed of termination, e.g. by experiments.

Our framework of properties is inspired by the one-way flow model that is introduced by Bala and Goyal (2000a). Besides the one-way flow model, Bala and Goyal (2000a) introduced another model, called the two-way flow model. The only difference between the one-way and the two-way flow model is that in the latter, profits flow in both directions of the links. Unfortunately, our results do not apply to the two-way flow model, since one of our properties, **BG**, is not satisfied here. To show this, consider the network $g = \{(j, i)\}$ where (j, i) is beneficial. Now consider the network $g' = g + (i, j)$. Here, agent i also observes j via (i, j), which implies that his own link (j, i) is not beneficial in g'. A framework of payoff properties that does cover the two-way flow model is examined by Derks and Tennekes (2008a).

Some properties that we introduced (**DE**, **BT**, and **BF**) are based on the assumption that indirect connections are not always worse than direct connections. In other words, these properties rule out information decay. The one-way flow model with decay can be realistic in the context of social networks, e.g. in friendship networks where a friend is more valuable than a friend of a friend. Further, our framework of payoff properties (especially **NA**) relies on the fact that links are perfectly reliable. Several models with imperfectly reliable links are studied in literature. It might be interesting to develop frameworks of axiomatic properties for these models.

The architecture of Nash networks is strongly related with the class of payoff functions. In this paper, proper networks play a prominent role. We proved existence of proper local- and global-Nash, and our dynamic procedure terminates at a proper network. Observe that proper networks also turn out to be essential in the one-way flow model. In fact, all Nash network characterizations provided by Galeotti (2006) are proper networks when link costs are (owner-)homogeneous. In Example 9 we showed that for our property framework, (strict) Nash networks can have other architectures as well. It would be interesting to examine which architectures are supported by other classes of payoff functions.

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