An axiomatic approach covering the two-way flow model of network formation

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Abstract

We provide a framework of axiomatic payoff properties such that two-way flow payoff functions satisfy these properties and such that we are able to show the existence of Nash networks.

Keywords: Non-cooperative Games, Network Formation, Axiomaization, Payoff functions.
JEL classification: C72, D85

1 Introduction

Consider a group of agents who want to share profits, for instance valuable information. The agents are able to link with other agents and thus form a network. By this network they can share profits. Each agent pays costs for each link that he formed and receives profits for being connected to other agents.

In this paper we study a model of unilateral network formation that is introduced by Bala and Goyal (2000). In this model, an agent can form a link unilaterally, i.e. without consent of the other agent. On the basis of the formed network, each agent gets a payoff, which consists of a costs and a profits part. Each agent pays some costs for each link that he forms. For the profits part, two different settings are considered, the one-way and the two-way flow model.

In the one-way flow model, a link that agent i forms is represented by an arc poining at i. Here, agent i receives a certain profit from being connected to agent j, which is the case if and only if a directed path from j to i exists. In other words, the profits flow along the direction of the arcs. In Figure 1a an example of a one-way flow network is given. Here, agent 1 is not connected to anyone and agents 2 and 3 are only connected to agent 1.

In the two-way flow model, a link that agent i forms is depicted as a line that is cut by a short line next to agent i. Agent i receives profits from being connected to agent j, if and only if an undirected path exist between them. In

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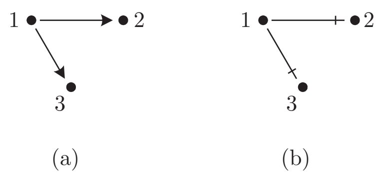


Figure 1: Examples of a one-way flow network (a), and a two-way flow network (b).

the two-way flow network depicted in Figure 1b, agents 1, 2 and 3 are connected to each other. Thus, the difference between the one-way and the two-way flow model is that in the latter, profits can flow in both directions of a link.

For the one-way flow model, Bala and Goyal (2000) prove the existence of Nash networks for payoff functions where link costs and profits are homogeneous, i.e. all links are equally expensive and all agents have the same profits. Furthermore, they characterize the architecture of these Nash networks. Galeotti (2006) extended the one-way flow model, by allowing link costs and profits to be hetereogeneous. He charaterizes (strict) Nash networks for various settings of heterogeneity. The existence of Nash networks for payoff functions with heterogeneous profits and owner-homogeneous link costs, i.e. all link costs are equally expensive with respect to the agent who forms them, has been proved simultaneously by Derks et al. (2008a) and by Billand et al. (2007). A short and elementary proof based on the former has been provided by Derks and Tennekes (2008b).

The two-way flow model has first been studied by Bala and Goyal (2000) and has then been extended by Galeotti et al. (2006) and Haller et al. (2007). Bala and Goyal (2000) prove the existence of Nash networks and characterize the architecture of these networks for payoff functions with homogeneous link costs and profits. Galeotti et al. (2006) extend this model by introducing heterogeneous link costs and profits. They charaterize the architecture of (strict) Nash networks. The existence of Nash network for payoff functions with heterogeneous profits and homogeneous link costs has been proved by Haller et al. (2007).

A model of unilateral network formation that covers the one-way flow model, has been studied by Derks et al. (2008b). They develop axiomatic properties for payoff function in such a way that they are intuitive and that they imply the existence of Nash networks. A full characterization of one-way flow payoff functions that satisfy these properties has been provided by Derks and Tennekes (2008a). They show that all one-way flow payoff functions with owner-homogeneous link costs satisfy these properties, as well as a subset of payoff functions with heterogeneous profits and link costs.

However, the axiomatic framework by Derks et al. (2008b) does not imply all two-way flow payoff functions. In this paper we develop axiomatic payoff properties that are intuitive in the context of the two-way flow model. With these payoff properties we prove the existence of Nash networks. We do this by generalizing the short and elementary proof by Haller et al. (2007). We give a full characterization of the two-way flow payoff functions that satisfy our properties.

We show that there are payoff functions with negative profits that satisfy these properties. Furthermore, we show that the links can be divided in two groups with respect to the owner: one with unaffordable links, and one with equally expensive links. Therefore, the formed network will only consist of affordable links. Hence we have a nice enhancement of the two-way flow model, namely where agents are restricted to form a specific set of links, i.e. the affordable links.

Furthermore, we provide examples of payoff functions that fall outside the scope of the two-way flow model, while they satisfy all our properties. These functions also take other properties of the network architecture into account, for instance the set of agents that are contained in a cycle.

2 Model

Let N be a finite set of agents. A link from agent j to i is denoted as (j,i). A *network* is defined as a set of links. Formally, a network is defined by $g \subseteq N \times N$ on the fixed set of agents N where loops are not allowed, i.e. $(i,i) \notin g$ for all $i \in N$. Let $\mathcal{G}$ be the set of all these networks.

For convenience we will use the symbol '+' for the union of two networks as well as for the union of a network with a single link, e.g. $g \cup g' \cup \{(j,i)\}$ equals $g + g' + (j,i)$.

For each agent i, let $\pi_i : \mathcal{G} \to \mathbb{R}$ be a payoff function. In this paper we introduce axiomatic properties for payoff functions. These properties are based on the payoff functions of the one-way flow model studied by Bala and Goyal (2000) and generalized by Galeotti (2006). We will refer to such specifif payoff functions as B&G-2 functions.

Before defining the B&G-2 functions, we need the following notation. Let $N_i(g)$ be the set of agents that i *observes* in g, i.e. the set of agents from whom an directed path to i exists in g and let $N_i^d(g)$ be the set of agents from whom a link pointing at i exists in g.

Payoff function $\pi_i(g)$ is called a B&G-2 function if

$$\pi_i(g) = v_i(N_i(\bar{g})) - c_i(N_i^d(g)) \tag{1}$$

where $\bar{g}$ is defined as the undirected version of network g, i.e. $\bar{g} = \{(j,i) : (j,i) \in g \text{ or } (i,j) \in g\}$, and where v_{ij} are the *profits* that i receives from being connected to j, which is the case if an undirected path between them exists in g, and where c_{ij} are the link costs. Here, we use the following shorthand notation: $v_i(S) = \sum_{j \in S} v_{ij}$ and $c_i(S) = \sum_{j \in S} c_{ij}$.

We say that link costs are homogeneous if there is a constant c with $c_{ij} = c$ for all $i, j \in N$. We say that link costs are owner-homogeneous if for each agent i there is a constant c_i with $c_{ij} = c_i$ for all $j \in N$. Otherwise, the link costs are heterogeneous. Analog definitions apply to the profits.

Existence of Nash networks where link costs and profits are homogeneous, is proven by Bala and Goyal (2000). Haller et al. (2007) show that Nash networks also exist when profits are heterogeneous. Furthermore, they provide a counterexample where Nash networks do not exist when link costs are heterogeneous.

The following payoff functions are called one-way flow payoff functions (refered to as B&G-1 functions).

$$\pi_i(g) = v_i(N_i(g)) - c_i(N_i^d(g)) \tag{2}$$

The difference between B&G-1 and B&G-2 functions is that in the one-way flow case an agent i only receives profits v_{ij} if $j \in N_i(g)$, i.e. if a direct path exists in g from j to i, while in the two-way flow case an agent i receives profits v_{ij} if $j \in N_i(\bar{g})$, i.e. if an undirected path exists between j and i in network g.

We define an *action* of agent i as a set of agents, denoted as $S_i \subseteq N \setminus \{i\}$. The network, after i chooses to link with the agents in S_i, is described by

$$g_{-i} + \{(j,i) : j \in S_i\}.$$

Here, $g_{-i} = g \setminus g_i$ is the network g with all i's links removed. The union of all the actions of all agents in N define the outcome network.

An action S_i of agent i is called a *best response* if

$$\pi_i\big(g_{-i} + \{(j,i) : j \in S_i\}\big) \geq \pi_i\big(g_{-i} + \{(j,i) : j \in T_i\}\big),$$

for all actions T_i. A network g is called a *Nash network* if $N_i^d(g)$ is a best response for all $i \in N$.

3 Axiomatization

In this section, we develop intuitive axiomatic payoff properties for the two-way flow model, in such away that we are able to prove the existence of Nash networks. For this purpose, we will use the constructive proof provided by Haller et al. (2007). This proof shows that Nash networks exist for games with homogeneous link costs and heterogeneous profits. We develop axiomatic payoff properties such that this proof holds for a more general class of payoff functions.

The first property that we introduce, disjoint additivity, is also used in our framework in chapter 4.

Property *DA* We say that a payoff function π is *disjoint additive* (*DA* for short), if for each two networks g and g', disjoint w.r.t. an agent i, we have

$$\pi_i(g + g') = \pi_i(g) + \pi_i(g').$$

Recall from chapter 4 that link (j,i) be called *profitable* if:

$$\pi_i(g) \geq \pi_i(g - (j,i)) \quad \text{if } (j,i) \in g,$$

and

$$\pi_i(g + (j,i)) \geq \pi_i(g) \quad \text{if } (j,i) \notin g.$$

Further, recall that a link (j,i) is called *beneficial* in network g whenever

$$\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i}),$$

where $g_{-ij} = g^j_{-i} + (j,i)$. Thus, the influence of a single link is measured in a network without own links.

The following result is analog to Lemma **??**.

Lemma 1 *If network g is minimal, and π is disjoint additive, then profitability and beneficiality are equivalent notions, more specifically,*

$$\pi_i(g) - \pi_i(g - (j,i)) = \pi_i(g_{-ij}) - \pi_i(g^j_{-i}) \quad \textit{for all } (j,i) \in g \tag{3}$$

$$\pi_i(g + (j,i)) - \pi_i(g) = \pi_i(g_{-ij}) - \pi_i(g^j_{-i}) \quad \textit{for all } (j,i) \notin g \textit{ where } j \notin N^u_i(g) \tag{4}$$

Proof. First, let $(j,i) \in g$. Since g is minimal, g_{-ij} and $g - g_{-ij}$ are i-disjoint. Therefore, by *DA* we have

$$\pi_i(g) = \pi_i(g_{-ij}) + \pi_i(g - g_{-ij}). \tag{5}$$

Since g is minimal, $g - (j,i)$ is also minimal. Hence by *DA* we obtain

$$\pi_i(g - (j,i)) = \pi_i(g^j_{-i}) + \pi_i(g - g^j_{-i} - (j,i)). \tag{6}$$

Since $g - g_{-ij} = g - g^j_{-i} - (j,i)$, we obtain (3) from (5) and (6).

Now, let $(j,i) \notin g$. Since g is minimal, and $j \notin N^u_i(g)$, networks g^j_{-i} and $g - g^j_{-i}$ are i-disjoint. By *DA* we obtain

$$\pi_i(g) = \pi_i(g^j_{-i}) + \pi_i(g - g^j_{-i}). \tag{7}$$

We have $g + (j,i) = g_{-ij} + (g - g^j_{-i})$. Since $j \notin N^u_i(g)$, it follow that g_{-ij} and $g - g^j_{-i}$ are i-disjoint. Hence by *DA* we obtain

$$\pi_i(g + (j,i)) = \pi_i(g_{-ij}) + \pi_i(g - g^j_{-i}). \tag{8}$$

Hence, (4) follows from (7) and (8). □

The next property is proposed in chapter 4.

Property *NA* We say that π satisfies *NA* (*naturality*) if $\pi_i(g + (k,i)) \leq \pi_i(g)$ whenever there is a directed path from k to i in the network g.

According to this property, for each agent $k \in N_i(g)$ where link (k,i) is not yet contained in g, agent i's payoff will not increase after adding (k,i) to g. This property is very intuitive in the one-way flow case, because of the following. In the one-way flow network depicted in Figure 2a, there is a directed path from k to i. Agent i already observes k, so he will not be better off by forming the direct link (k, i). However, in the two-way flow case, agent i also observes k if an undirected path exists between them (see Figure 2b). Therefore we propose the following alternative.

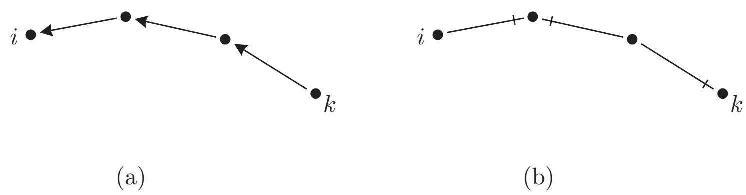


Figure 2: Agent i is connected to k in the one-way flow case (a) and in the two-way flow case (b).

Property *NA*2 We say that π satisfies $NA2$ (*naturality in the undirected case*) if $\pi_i(g + (k, i)) \leq \pi_i(g)$ whenever there is a undirected path from k to i in the network g.

Observe that by this property, minimal networks are preferred over non-minimal networks. More specifically, for any non-minimal network g, an agent exists who can play a good local response by deleting a redundant link.

In the following theorem we show that for payoff functions that satisfy DA and $NA2$, each minimal local-Nash network is also a global-Nash network.

Theorem 2 *Let π satisfy DA and NA2. Then each minimal local-Nash network is global-Nash.*

Proof. Let g be a minimal local-Nash networks. Suppose to the contrary that g is not global-Nash, say i can strictly improve in g. Let $S = N_i^d(g)$ be his current action, and let $\tilde{S}$ be a strictly improving action, such that $|\tilde{S} \setminus S|$ is as small as possible and such that among those, $|S \setminus \tilde{S}|$ is as small as possible. Let $\tilde{g}$ be the network obtained after i plays $\tilde{S}$.

Let $j \in \tilde{S} \setminus S$. Suppose that $j \in N_i^u(g)$. Then by $NA2$, the action $\tilde{S} - j$ is at least as good as $\tilde{S}$, with $|(\tilde{S} - j) \setminus S| < |\tilde{S} \setminus S|$. Hence we have a derived a contradiction.

Hence $j \notin N_i^u(g)$. By $NA2$ we may assume that g_{-ij} is i-disjoint from $\tilde{g} - g_{-ij}$. Hence by DA we obtain

$$\pi_i(\tilde{g}) = \pi_i(\tilde{g} - g_{-ij}) + \pi_i(g_{-ij}), \quad \text{and} \tag{9}$$

$$\pi_i(\tilde{g} - (j, i)) = \pi_i(\tilde{g} - (j, i) - g_{-i}^j) + \pi_i(g_{-i}^j). \tag{10}$$

Since g is local-Nash, we have $\pi_i(g + (j, i)) \leq \pi_i(g)$. By Lemma 1, it follows that $\pi_i(g_{-ij}) \leq \pi_i(g_{-i}^j)$. Hence by (9) and (10) we obtain $\pi_i(\tilde{g}) \leq \pi_i(\tilde{g} - (j, i))$.

Hence $\tilde{S} - j$ is at least as good as $\tilde{S}$, with $|(\tilde{S} - j) \setminus S| < |\tilde{S} \setminus S|$. This is a contradiction.

Hence we conclude that $\tilde{S} \subseteq S$.

Let $j \in S \setminus \tilde{S}$. Since g is minimal local-Nash, (j, i) is profitable and by Lemma 1 also beneficial. Since $\tilde{g} \subset g$, network $\tilde{g}$ is also minimal. Hence it follows that (j, i) is also profitable in $\tilde{g}$. Hence $S + j$ is at least as good as S, with $|S \setminus (\tilde{S} + j)| < |S \setminus \tilde{S}|$. This is a contradiction.

Hence we conclude that $\tilde{S} = S$, which contradicts $\tilde{S}$ being a strict improvement. Hence, g is global-Nash. □

The following property has been proposed in chapter 4.

Property *BG* Payoff function π satisfies *BG* (*beneficial growth*) if $\pi_i((g + (k, r))_{-ij}) \geq 0$ for each pair of agents $k,\ r$, whenever $\pi_i(g_{-ij}) \geq 0$.

However, it does not apply to all B&G-2 payoff functions. To see this, consider a network g where agent i has one link, say (j, i), by which he can observe all agents in a subset of agents S. Thus, the B&G-2 function is the following: $\pi_i(g_{-ij}) = v_i(S) - c_{ij}$. Now, let agent $r \in S$ add the link (i, r). Then, agent i can observe all agents in S by that link, which makes his own link redundant. Hence $\pi_i((g + (k, r))_{-ij}) < \pi_i(g_{-ij})$ if the link cost c_{ij} is strictly positive, and therefore it may be that $\pi_i((g + (k, r))_{-ij}) < 0$ while $\pi_i(g_{-ij}) \geq 0$. We can fix this by the following refinement.

Property *BG*2 Payoff function π satisfies *BG*2 (*beneficial growth in the undirected case*) if $\pi_i((g+(k, r))_{-ij}) \geq \pi_i((g+(k, r))^j_{-i})$ for each pair of agents $k,\ r$, whenever $g + (k, r)$ is minimal and $\pi_i(g_{-ij}) \geq \pi_i(g^j_{-i})$.

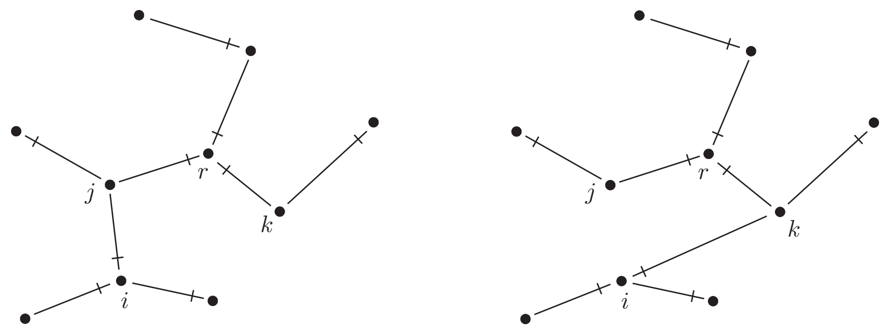
Thus, a link remains beneficial if another agent adds a link such that the obtained network is minimal. The intuition behind this property is that in a minimal network, at most one undirected path exists between each pair of agents. Hence, the set of agents observed by agent i via link (j, i) in a minimal network g, is also observed via (j, i) in the minimal network $g + (k, r)$.

The last property that we introduce is very demanding.

Property *RP* Payoff function π satisfies *RP* (*replacement*) if the following holds. Let g be a minimal network where $(j, i) \in g$. If (j, i) is beneficial, then $\pi_i(g_{-ik}) \leq \pi_i(g_{-ij})$, for each agent $k \in \mathrm{Car}(g^j_{-i})$.

Thus, if a link is beneficial, then a replacement by another link in the same component is not an improving action. The intuition behind this property is the following. Our aim is to design a property such that beneficial links in a minimal network remain beneficial after a replacement. We do not want to encourage replacements inside a component, because of the following situation. Consider the network g depicted in Figure 3(a). Consider the replacement of (j, i) by (k, i) where the obtained network g' is depicted in Figure 3(b). Now focus on link (j, r). In network g, agent r observes 5 agents via this link, including agent i, whereas in network g', he only observes 2 of these agents. When we consider B&G-2 functions, this implies that link (j, r) can be beneficial in g

but not anymore in g'. By this argument, we discourage replacements inside a component with the property RP.



(a) Network g (b) Network g'

Figure 3: A replacement

Now we generalize the result of Haller et al. (2007), which is that Nash networks exists for B&G-2 functions with heterogeneous profits and homogeneous link costs. We reconstruct their constructive proof where we implement the payoff properties DA, $NA2$, $BG2$ and RP.

Theorem 3 *Let π be a payoff function that satisfies DA, NA2, BG2 and RP. Then a minimal, local-Nash network exists.*

Proof. By iteration, we construct a minimal network which is local-Nash, beginning with the empty network. Observe that the empty network is minimal and beneficial.

Let g be any minimal and beneficial network and not local-Nash, say i can improve.

By Lemma 1 it follows that each link in g is profitable, and therefore i does not gain from removing a link.

Because of RP, player i does not strictly prefer to replace link (j, i) with (k, i), where $k \in \mathrm{Car}(g^j_{-i})$.

Consequently, i agent is better off by adding a new link, say (j, i). Let the obtained network be $g' = g + (j, i)$. By $NA2$, we may assume that $j \notin N_i(g)$. Hence g' is also minimal. Since (j, i) is profitable in g', it follows by DA and Lemma 1 that link (j, i) is also beneficial in g'. Further, the other links in g' are beneficial by $BG2$. Hence, g' is beneficial.

Since g' is minimal and beneficial we can repeat this step if g' is not local-Nash, until we obtain a local-Nash network. At each iteration, a link is added, and therefore, the network grows. Since a minimal network of n agents has at most $n-1$ links, we know that in finitely many iterations we obtain a local-Nash network. □

Since the network that we obtained in this proof is minimal, we obtain the following corollary from Theorem 2.

Corollary 4 *For any payoff function that satisfies DA, NA2, BG2 and RP, a global-Nash network exists. Specifically, a minimal, global-Nash network.*

4 Property independence

The following payoff function satisfies all properties, except DA:

$$\pi_i(g) = |N_i(\bar{g})|^2 \tag{11}$$

Property DA is not satisfied, because for any two i-disjoint networks g and g' where $i \in \mathrm{Car}(g) \cap \mathrm{Car}(g')$, we have $|N_i(\bar{g})|^2 + |N_i(\bar{g}')|^2 < |N_i(\bar{g} \cup \bar{g}')|^2$. The properties $NA2$, $BG2$ and RP are clearly satisfied.

The following payoff function satisfies all properties, except $NA2$:

$$\pi_i(g) = |N_i^d(g)| \tag{12}$$

Property $NA2$ is not satisfied, because for any network g where (k, i) is not present, we have $\pi_i(g + (k, i)) = \pi_i(g) + 1$, hence also if a path exists between k and i. Property DA is clearly satisfied. Properties $BG2$ and RP are satisfied because $\pi_i(g_{-ij}) = \pi_i(g_{-ik})$ for any network g and any two agents j and k.

The following payoff function satisfies all properties, except $BG2$:

$$\pi_i(g) = -|N_i(\bar{g})| \tag{13}$$

Property $BG2$ is not satisfied here, because two components may be connected to each other by the addition of a link, such that the obtained network is minimal and such that agent i observes strictly more agents. Properties DA, $NA2$ and RP are clearly satisfied.

The following payoff function satisfies all properties, except RP:

$$\pi_i(g) = \sum_{\text{component } g' \subseteq g} \begin{cases} |N_j^d(g')| & \text{if } N_i^d(g') = \{j\}; \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

This payoff function does not satisfy RP, because in a connected and minimal network, where agent k has more links than j, we have that $\pi_i(g_{-ij}) < \pi_i(g_{-ik})$. Property DA is satisfied because this payoff function is built up componentwise. Property $NA2$ is satisfied because if agent i has two links or more in one component, then his payoff is lower than any other payoff. Property $BG2$ is satisfied, because $\pi_i(g_{-i}) = 0$ for all networks g, and all other payoffs are non-negative.

5 Characterization

Now we would like to know to what extend we generalized the result of Haller et al. (2007) for B&G-2 functions. In other words, the question is which B&G-2 functions satisfy the properties DA, $NA2$, $BG2$ and RP, and therefore imply the existence of local- and global-Nash networks.

It can be easily seen that $NA2$ is satisfied whenever $c_{ij} \geq 0$. We will refer to this non-negativity as NNC (non-negative link costs). Property DA is clearly satisfied by all B&G-2 functions. The following lemma characterizes the set of B&G-2 functions that satisfy $BG2$.

Lemma 5 *Let π be a B&G-2 payoff function. Then π satisfies BG2 if and only if the following property holds:*

PBG If $c_{ij} \leq v_i(S)$ for agent set $S \subset N$ and agent $j \in S$, then $c_{ij} \leq v_i(S')$ for all $S' \supset S$.

Proof. First suppose that PBG does not hold. Then a set $S \subset N$, a set $S' \supset S$, and an agent $j \in S$ exist such that $c_{ij} \leq v_i(S)$ and $c_{ij} > v_i(S')$. Consider a minimal network g where i has one incoming link, (j, i), and no outgoing links. Thus, $\pi_i(g_{-i}) = 0$. Furthermore, let all agent in S form a component and all agents in $S' \setminus S$ form a component. Let k be an agent in S and let r be an agent in S'. Since $(j, i) \in g$, and $j \in S$ we have $\pi_i(g) = v_i(S) - c_{ij} \geq 0$, and $\pi_i(g + (k, r)) = v_i(S') - c_{ij} < 0$. Hence, $BG2$ is not satisfied.

Now suppose that $BG2$ does not hold. Then a network g exists such that link $(k, r) \notin g$, network $g + (k, r)$ is minimal, $\pi_i(g_{-ij}) \geq \pi_i(g_{-i})$, and $\pi_i((g + (k, r))_{-ij}) < \pi_i((g + (k, r))_{-i})$. Let S be the set of agents that i observes in g using link (j, i). For network $g+(k, r)$, let S' be the set of agents that i observes using (j, i). Notice that since g and $g + (k, r)$ are both minimal networks, i uniquely observes S and S' via (j, i). Hence $\pi_i(g_{-i}) = \pi_i((g + (k, r))_{-i}) = 0$. Since $BG2$ does not hold, S and S' can not be equal. Therefore, $S' \supset S$. We have $0 \leq \pi_i(g_{-ij}) - \pi_i(g_{-i}) = v_i(S) - c_{ij}$, and $0 > \pi_i((g + (k, r))_{-ij}) - \pi_i((g + (k, r))_{-i}) = v_i(S') - c_{ij}$. Hence we have obtained $c_{ij} \leq v_i(S)$ and $c_{ij} > v_i(S')$. Thus we conclude that PBG is not satisfied. □

Notice the similarity between the characterization of $BG2$ with respect to B&G-2 functions and the characterization of BG with respect to B&G-1 functions.

The next lemma characterizes what B&G-2 functions satisfy property RP.

Lemma 6 *Let π be a B&G-2 payoff function. Then π satisfies RP if and only if the following property holds:*

PRP If $c_{ij} \leq v_i(S)$ for agent set $S \subseteq N$ and agent $j \in S$, then for each $k \in S$ it holds that either $c_{ik} = c_{ij}$ or $c_{ik} > v_i(S)$.

Proof. Suppose that PRP is not satisfied. Then a set $S \subset N$ and agents $j, k \in S$ exist where $c_{ij} \leq v_i(S)$ and $c_{ik} \neq c_{ij}$ and $c_{ik} \leq v_i(S)$. Without loss of generality we may assume that $c_{ik} < c_{ij}$. Consider a network g where i observes set S in both g_{-ij} and g_{-ik}. Hence, $\pi_i(g_{-ij}) = v_i(S) - c_{ij} < v_i(S) - c_{ik} = \pi_i(g_{-ik})$, and therefore, RP is not satisfied.

Now suppose that RP is not satisfied. Then a minimal network g exists, where $(j, i) \in g$, where $\pi_i(g_{-ij}) \geq \pi_i(g_{-i})$, where $k \in g^j_{-i}$ and where $\pi_i(g_{-ik}) > \pi_i(g_{-ij})$. Let S be the set of agents that i observes by (j, i), i.e. $S = \mathrm{Car}(g^j_{-i})$. Since g is a minimal network, it follows that $0 \geq \pi_i(g_{-ij}) - \pi_i(g_{-i}) = v_i(S) - c_{ij}$, and hence that $c_{ij} \leq v_i(S)$. Since $0 < \pi_i(g_{-ij}) - \pi_i(g_{-ik}) = c_{ik} - c_{ij}$, it follows that $c_{ik} < c_{ij}$. Therefore we have that $c_{ik} \leq v_i(S)$ and $c_{ik} \neq c_{ij}$; a contradiction with PRP. □

Hence we derive the following theorem from Corollary 4.

Theorem 7 *Let π be a B&G-2 payoff function. Then a global-Nash network exists when*

(NNC)	$c_{ij} \geq 0$	$\forall j \in N$, *and*

if $c_{ij} \leq v_i(S)$ for agent set $S \subseteq N$ and agent $j \in S$ then

(PBG)	$c_{ij} \leq v_i(S')$	$\forall S' \supseteq S$, *and*
(PRP)	$c_{ik} = c_{ij}$ *or* $c_{ik} > v_i(S)$	$\forall k \in S$

Hence, there exists B&G-2 functions with negative profits for which global-Nash networks exist. The following theorem shows that PBG and PRP imply links can be divided in two groups: one with affordable, owner-homogeneous links, and one with unaffordable links (i.e. $c_{ij} > v_i(N)$).

Theorem 8 *Let π be a B&G-2 payoff function that satisfies PBG and PRP. If $c_{ij} \leq v_i(S)$ for a set S and an agent $j \in S$, then for each $k \in N$ either $c_{ik} = c_{ij}$ or $c_{ik} > v_i(N)$.*

Proof. By PBG it follows directly that if $c_{ij} \leq v_i(S)$ for some agent j and set S, then $c_{ij} \leq v_i(N)$. By PRP this implies that for each $k \in N$, either $c_{ik} = c_{ij}$ or $c_{ik} > v_i(N)$. □

In Figure 4 we show the characterization of B&G-2 functions graphically. Each area corresponds to a specific subset of B&G-2 functions. Area 8 corresponds to B&G-2 functions that satisfy NNC, PBG and PRP. Recall that each B&G-2 function trivially satisfies DA. Therefore, local- and global-Nash networks exist for each game with a B&G-2 function in area 8. For each area, an example of a B&G-2 function is given in Table 1. In these examples, 3 agents are involved, except for area 3 where 4 agents are involved[1].

[1]It can be verified that an example with 3 agents for area 3 does not exist.

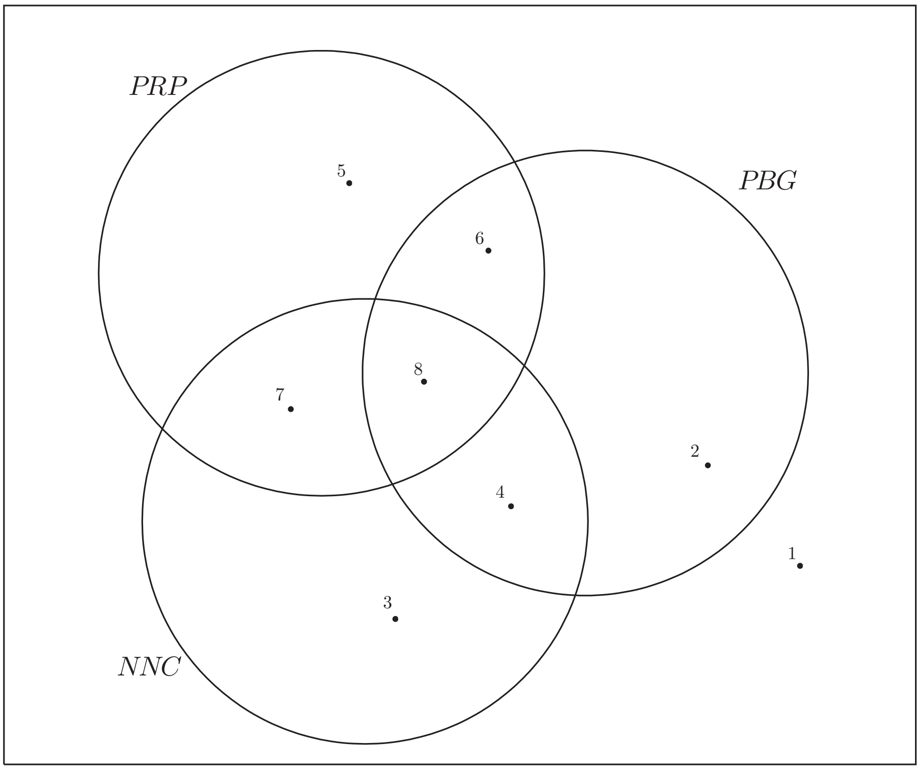


Figure 4: The characterization of B&G-2 functions.

Table 1: Characterization of the eight area's in Figure 4

Area	Properties			Example					
	PRP	*NNC*	*PBG*	c_{ij}	c_{ik}	$c_{i\ell}$	v_{ij}	v_{ik}	$v_{i\ell}$
1	n	n	n	-1	0		0	-2	
2	n	n	y	-1	3		2	1	
3	n	y	n	0	1	0	1	1	-1
4	n	y	y	0	1		0	1	
5	y	n	n	-1	-1		0	-2	
6	y	n	y	-2	-2		0	1	
7	y	y	n	2	2		2	-1	
8	y	y	y	2	2		3	-1	

We cannot draw conclusions by comparing this characterization with the characterization of B&G-1 functions in chapter 6, because the profits are used differently in B&G-1 and B&G-2 functions. We can just observe that in both the one- and two-way flow model, the existence of global-Nash networks is guaranteed for games with owner-homogeneous link costs, and non-negative link costs and profits.

By Theorem 7, we know that for the two way flow model, local- and global-Nash networks also exist for games with negative profits and owner-homogeneous link costs. Recall from chapter 2 that this result does not hold for B&G-1 functions (see Example **??**).

Furthermore, we know that local- and global-Nash networks exist for games with two groups of links: the first group contains links with owner-homogeneous costs, and the second group contains unaffordable links. From this observation we obtain the following enhancement of the two-way flow model. Each agent is restricted to form a subset of links. Formally, define an action of agent i as $S_i \subseteq M_i$, where $M_i \subseteq N \setminus \{i\}$. Thus, agent i can only link with agents in M_i. Clearly, this enhanced model is equivalent with the original model where each link (j, i) is affordable if $j \in M_i$ and unaffordable if $j \notin M_i$.

The following payoff functions show that our axiomatic approach yields a generalization of B&G-2 functions.

$$\pi_i(g) = |SP_i(\bar{g})| - |N_i^d(g)|; \tag{15}$$

$$\pi_i(g) = |C(\bar{g}) \cap N_i(\bar{g})| - |N_i^d(g)|, \tag{16}$$

where $SP_i(\bar{g})$ is the number of spokes that i observes in $\bar{g}$ and $C(\bar{g})$ is the number of agents that are contained in a directed cycle in $\bar{g}$. These two payoff functions satisfy all properties, i.e. DA, $NA2$, RP and $BG2$. They fall outside the scope of B&G-2 functions, because B&G-2 functions only take the sets $N_i(g)$ and $N_i^d(g)$ into account, while the payoff functions (15) and (16) also take other properties of the network architecture into account.

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